Critical remarks on Sokolov's theory of radiating electron (extended toward quantum mechanics)

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Abstract

In the article [2] I.V. Sokolov, *Renormalization of the Lorentz-Abraham-Dirac equation for radiation reaction force in classical electrodynamics*, JETP, 109(2), 2009 there was proposed a so-called renormalization of the Lorentz-Abraham-Dirac equation (LAD). The latter describes the motion of an electron in electromagnetic field in consideration of the radiation reaction. In many articles of I.V. Sokolov and his colleagues this model is promoted as more appropriate one when an electron is under the effect of a super-strong laser field. In present article it is shown that such pretensions are baseless and physical importance of this model is vastly overestimated. Sokolov's equation suffers from numerous drawbacks arising from the fact that it represents not modified but distorted LAD. In particular, superluminal paradoxes arise in the framework of this model.

§ 1. Sokolov's equation

The dynamical equation of an electron in electromagnetic field looks as follows:

$$m\ddot{x}^i = \frac{e}{c}F^{ij}\dot{x}_j \tag{0}$$

A point from above everywhere denotes a derivative with respect to the proper time τ. In the Lorentz-Abraham-Dirac equation (LAD) [1] the radiation reaction had been taken into account:

$$m\ddot{x}^i = \frac{e}{c}F^{ij}\dot{x}_j + m\tau_0\dddot{x}^i + \frac{m\tau_0\ddot{x}^2}{c^2}\dot{x}^i \tag{1}$$

where $\tau_0 = 2e^2/(3mc^3) \approx 6.2 \cdot 10^{-24}$ sec. LAD is equivalent to the following system:

$$\dot{p}^i = \frac{e}{c}F^{ij}\dot{x}_j - \dot{p}^i_{rad} \qquad \dot{x}^i = \frac{1}{m}p^i + \dot{x}^i_{rad} \tag{2}$$

where the radiation amendments are expressed by the following formulas:

$$\dot{p}^i_{rad} = -\frac{m\tau_0\ddot{x}^2}{c^2}\dot{x}^i \qquad \dot{x}^i_{rad} = \tau_0\ddot{x}^i \tag{3}$$

Thus, in "phase coordinates" LAD can be rewritten as the system:

$$\dot{p}^i = \frac{e}{c}F^{ij}\dot{x}_j + \frac{m\tau_0\ddot{x}^2}{c^2}\dot{x}^i \qquad \dot{x}^i = \frac{1}{m}p^i + \tau_0\ddot{x}^i \tag{4}$$

The following two relativistic identities play the key part further:

$$(V) \quad \dot{x}^2 = c^2 \ , \qquad (P) \quad p^2 = m^2c^2$$

where $\dot{x}^2 = \dot{x}^i\dot{x}_i$ and $p^2 = p^ip_i$ with metric tensor $g_{ij} = diag(1, -1, -1, -1)$.

(V) condition holds for any solutions of (1) satisfying $\dot{x}^i(0)\dot{x}_i(0) = c^2$ and $\dot{x}^i(0)\ddot{x}_i(0) = 0$. Indeed, for such a solution $x^i(\tau)$ the following ODE with respect to $z = \dot{x}^2/c^2 - 1$

$$\ddot{z} = \frac{1}{\tau_0}\dot{z} - \frac{2}{c^2}\ddot{x}^2 z$$

has the only solution $z(\tau) \equiv 0$ satisfying $z(0) = 0$ and $\dot{z}(0) = 0$. This equation has been derived from (1) by multiplication and contraction by $\dot{x}_i$.

But (P) condition is insignificantly violated in the case of LAD (§3). In [2] there was proposed an equations system for which (P) allegedly holds but violations of (V) are allowed. With this regard system (2) with arbitrary x^i_{rad} and p^i_{rad} was considered under the condition (17) [2]:

$$\frac{e}{c}p^iF_{ij}\dot{x}^j_{rad} = p_i\dot{p}^i_{rad} \tag{5}$$

Easy to verify that (5) is equivalent to $\dot{p}^ip_i = 0 \Leftrightarrow p^2 = const$. One of the systems (2) satisfying (5) is determined by the conditions (18) [2]:

$$\dot{p}^i_{rad} = \frac{I}{mc^2}p^i \ , \quad \dot{x}^i_{rad} = \frac{\tau_0}{m}\frac{I}{I_E}f^i_L \ , \quad I_E = -\frac{\tau_0}{m}f^2_L \ , \quad f^i_L = \frac{e}{mc}F^{ij}p_j \tag{6}$$

where f^i_L and I_E are called the Lorentz force and the intensity of electric dipole radiation, undefined quantity I of power dimension is called the intensity of radiation. Note that $p^i \neq m\dot{x}^i$ in the case of LAD (and Sokolov's equation too). Equations (2) and (6) result in:

$$\dot{p}^i = \frac{e}{mc}F^{ij}p_j - \frac{I}{mc^2}p^i + \frac{\tau_0e^2}{m^2c^2}\frac{I}{I_E}F^{ik}F_{kj}p^j \tag{7}$$

Note that equation (8) is insufficient for describing the dynamics because now $p^i \neq m\dot{x}^i$. The author declared on the possibility of using arbitrary I including random functions [2]. Assuming $I = I_E$ the following system (19) [2] is obtained from (2) and (6) :

$$\dot{p}^i = \frac{e}{c}F^{ij}\dot{x}_j + \frac{\tau_0f^2_L}{m^2c^2}p^i \qquad \dot{x}^i = \frac{1}{m}p^i + \frac{\tau_0}{m}f^i_L \tag{8}$$

In §9 [2] equations (8) are rewritten in the 3-dimensional form (21),(22),(23) [2]. These latter as well as (8) are called Sokolov's equation(s), SEQ in what follows. The author calls it renormalized LAD or modified LAD.

According to §5 [2], the word "renormalization" means a redefinition of the operator $\hat{m}$: $\dot{x}^j \longmapsto p^j$ which is determined by the dynamics equations. For LAD and SEQ these operators look respectively:

$$\hat{m}^i_j = m\Big(1 - \tau_0 \frac{d}{d\tau}\Big)\delta^i_j \qquad\qquad (\hat{m}^{-1})^i_j = \frac{1}{m}\Big(\delta^i_j + \frac{\tau_0 e}{mc}\frac{I}{I_E}F^{i\cdot}_{j}\Big)$$

Obviously, the second "operator" has nothing common with the inverse first one. Moreover, it is not a linear operator generally speaking. Indeed, I_E depends on p^i (6). If I is not proportional to I_E, then $\hat{m}$ is not a linear operator and, hence, may not be considered as an operator at all. The author claims that (6) could have a quantum interpretation by means of using random functions I (see §8 [2]). But then the notion of operator $\hat{m}$ will lose a sense.

The following assertion from abstract of [2] points to Dirac as the fountainhead of SEQ: "*The present analysis makes use of an idea that was put forward by Dirac in [7], but was never developed*". Any grounds for this opinion are absent in [1]. Neither links nor relevant quotes from Dirac are given in [2]. A reader is proposed to believe that somewhere and once Dirac said something that sanctifies this result? An example is the following quote from §6 [2]: "*The arbitrariness in the expression for momentum (i.e., in mass operator renormalization) pointed out by Dirac lies in the choice of which of these identities should be dropped ...*". Which work by Dirac does point out this? Nothing similar was found in [1] but no other links to Dirac's works are given in [2].

For the same problem the equation of Landau-Lifshitz (LL) was proposed in (76, 3) [14]:

$$m\ddot{x}^i = \frac{e}{c}F^{ij}\dot{x}_j + \frac{e}{c}\tau_0\frac{\partial F^{ik}}{\partial x_l}\dot{x}_k\dot{x}_l - \frac{3c}{2}\tau_0^2 F^{ij}F^{k}_{\cdot j}\dot{x}_k + \frac{3}{2c}\tau_0^2 F^{jk}F_j^{\cdot l}\dot{x}_k\dot{x}_l\dot{x}^i \tag{9}$$

Differentiating the 2nd equation (8) one obtains the following equation:

$$\begin{aligned} m\ddot{x}^i = {} & \tfrac{e}{c}F^{ij}\dot{x}_j + \tfrac{e}{c}\tau_0\tfrac{\partial F^{ik}}{\partial x_l}\big(\dot{x}_k - \tfrac{f_{L,k}}{m}\tau_0\big)\big(\dot{x}_l - \tfrac{f_{L,l}}{m}\tau_0\big) - \tfrac{3c}{2}\tau_0^2 F^{ij}F^{k}_{\cdot j}\big(\dot{x}_k - \tfrac{f_{L,k}}{m}\tau_0\big) + \\ & + \tfrac{3}{2c}\tau_0^2 F^{jk}F_j^{\cdot l}\big(\dot{x}_k - \tfrac{f_{L,k}}{m}\tau_0\big)\big(\dot{x}_l - \tfrac{f_{L,l}}{m}\tau_0\big)\big(\dot{x}^i - \tfrac{f_L^i}{m}\tau_0\big) + \tau_0^3(\ldots) \end{aligned} \tag{10}$$

Neglecting $\tau_0^3(\ldots)$ and all the terms with $f_L\tau_0/m$ factors we would obtain (9). But this cannot be done in ultra-relativistic case for which, according to Sokolov, his equations are intended. Indeed, in such a case the radiation amendment $\dot{x}^i_{rad}$ may be comparable to $\dot{x}^i$ (2). As $\dot{x}_{rad} = f_L\tau_0/m$ (8) then the terms with $f_L\tau_0/m$ factors cannot be omitted. Further attempts to express f_L^i through x^i by using the 2nd equations (2) and (8) will cause only the "recursive calls" with increasingly complicated formulas. Hence, (10) cannot be reduced to an ODE of the 2nd order with respect to x^i having an explicitly defined right-hand part. Comparing (9) and (10) easy to see the SEQ significantly differs from LL.

Article [2] became a source for many other publications related to the dynamics of an electron in consideration of the radiation reaction. Amongst them are [3 - 9] where system

(8) is presented to be more successful compared to (4). The author decidedly declares on the advantages of his model basing himself, in fact, on subjective value judgments.

In [4] the following is asserted. "*In [24,25] is described a derivation of the modified LAD equation for electrons and account for radiation from electrons and the electron current in a plasma, in a self-consistent manner. It is shown that the new equation is free of runaway solutions, providing a different set of exact conservation laws.*"

In fact, SEQ is distorted but not modified LAD (§2). As shown in §3, the energy-momentum conservation law holds in the case of LAD and is violated by SEQ. The author didn't clarify what other conservation laws out of this "different set" he meant. With regards to the conservation of generalized momentum, in the article [3] there was pointed that it may take place only in a special case when $\mathbf{n} \cdot \nabla A^i = 0$ and $\mathbf{n} \cdot \mathbf{p} = 0$ for some direction $\mathbf{n}$. Obviously, such an electron motion is impossible in a laser field. So the mention of generalized momentum conservation plays a decorative part.

In [9] the following value judgment is presented like a confirmed and recognized fact. "*Another way to solve the problem was suggested by renormalization of LAD equation. The new equation better satisfies conservation laws for relativistic particles, suggest an efficient numerical scheme and allows for its extension toward the QED regime.*"

In fact, there are no serious grounds for the opinion that QED is well compatible with SEQ (§5). No information about experimental confirmations of this mathematical model was found in [2 - 10]. From the theoretical point Sokolov's result is questionable. Despite of that, now it holds pride of place among the classical results on this problem [11,12].

More sober estimations also exist. An example is §3.4 [13]. "*Since its inception, the Sokolov theory has gained significant attention (though it is still far from universally accepted). However, it should be noted that this theory too suffers a number of drawbacks, stemming from abandoning the normalization condition in (28).*"

Here the normalization condition is the identity $\dot{x}^2 = c^2$ (see (V) below (4)). The following conclusion from §3.4 [13] is mistaken. "*More recently, Sokolov [23] has introduced an equation that also departs radically from a conventional tenet of physics, in this case that the 4-momentum should be collinear with the 4-velocity*". Such a non-collinearity also takes place for LAD.

§ 2. Distortion of LAD

Formulas (6) may be obtained by means of only (P) condition without any other physical ideas. Indeed, taking LAD as a base from (3) it is seen that $\dot{p}^i_{rad} = Kp^i$ and $\dot{x}^i_{rad} = Mf^i_L$ for some quantities K and M. Then if (5) and (P) are assumed to be valid, easy to derive that

$$\dot{x}^i_{rad} = K\tau_0 \frac{c^2}{I_E} f^i_L$$

Since $[K] = sec^{-1}$ then $I = Kmc^2$ has power dimension (or the intensity of something). Expressing K through I formulas (6) will be derived instantly. Thus, SEQ with a quite arbitrary quantity I has been obtained. In particuar, may be not connected with this electron and the field. The physical emptiness of such a model is obvious.

(P) condition holds for any SEQ solution $x^i(\tau), p^i(\tau)$ satisfying $p^i(0)p_i(0) = m^2c^2$. Indeed, after multiplication and contraction by p_i the 1st equation (2) with (6) result in the following ODE with respect to $y = p^2 - m^2c^2$:

$$\dot{y} = -\frac{2I}{mc^2} y$$

The latter has the only solution $y(\tau) \equiv 0$ satisfying $y(0) = 0$. Easy to verify that this proof also works in the case when

$$\dot{p}^i = \frac{e}{c} F^{ij}\dot{x}_j + \frac{e\tau_0}{m^2c^3}\big(F^{jk}\ddot{x}_j p_k\big)p^i \qquad \dot{x}^i = \frac{1}{m}p^i + \tau_0 \ddot{x}^i \tag{11}$$

The 1st equation (11) has just the same form as the 1st one (8), if the intensity of electric dipole radiation is defined by the following formula:

$$I = I_E = -\frac{e\tau_0}{mc} F^{jk}\ddot{x}_j p_k$$

In the case $p^i = m\dot{x}^i$ this would give the same value as the 3rd formula in (6). Note that in the case under consideration the last formula (6) differs from the right-hand side of (0). Obviously, there are no physical reasons for considering the I_E formula assumed in (11) to be worse than the 3rd formula (6). Herewith the 2nd equation (11) looks more adequate taking into account the classical formula for the radiation reaction force (75,8 [14]).

Thus, equations system (11) represents another "renormalized LAD" with the same property with respect to (P) as that of SEQ. Namely, any (11) solution $x^i(\tau), p^i(\tau)$ satisfying the initial condition $p^i(0)p_i(0) = m^2c^2$ satisfies (P) at every moment τ. This example clearly shows that (P) condition in itself may not be considered to be a source of physically meaningful equations of the electron dynamics. Though this is not easy to discern in [2], no other sources were used there despite of the plenty of fuzzy reasonings that have no direct relation to the *real* procedure of SEQ obtaining.

Not difficult to dream up completely another system (2) which also provides solutions satisfying (P). Indeed, for some constant A consider the equations:

$$\dot{p}^i_{rad} = A\tau_0 \dot{p}_i \qquad \dot{x}^i_{rad} = -A\tau_0 \dot{x}_i \ , \qquad 0 < |A|\tau_0 < 1$$

Then from (2) there can be derived the following, attractively simple equation:

$$m\ddot{x}^i = \frac{e}{c(1+A\tau_0)^2} F^{ij}\dot{x}_j$$

Easy to verify that for all its solutions we have $p^2 = const$. If there is chosen a partial solution satisfying (P), then $\dot{x}^2 = c^2/(1+A\tau_0) \neq c^2$. This violation of (V) will be small for a small enough constant A that should be determined by experiments. What are the grounds to consider less suitable than (6)? They both base on (2), satisfy the condition $p^2 = const$ and weren't confirmed by experiments. Assuming $\tau_0 = 0$ both the systems reduce to equation (0). May one consider this equation constructed quite arbitrary as a "renormalization" of LAD, which is more rigorous and appropriate for describing electrons in super-strong laser fields?

Consider an unexpectedly simple way of obtaining SEQ directly from LAD for the case $I = I_E$. Let's express the right-hand sides of formulas (3) through the variables p^i assuming that $p^i = m\dot{x}^i$ and $\dot{p}^i = e/(mc)\cdot F^{ij}p_j$ (these equations would describe the motion without accounting the radiation force). Indeed, let $\dot{x}^i_{rad} = \tau_0\dot{p}^i/m = \tau_0/m \cdot f^i_L$. Since $\tau_0^2\ddot{x}^2 = \dot{x}^2_{rad}$ then $\dot{p}^i_{rad} = I_E/(mc^2)\cdot p^i$, where I_E is defined in (6). Then system (8) follows from (2). Thus, SEQ represents not renormalization but a distortion of LAD.

In [2] the author attempted to derive equation (7), which coincides with (5) [3], from the energy-momentum conservation. There was considered the case when an electron is absorbing n photons of a plane wave and is emitting a photon with higher energy, i.e., $p^i_1 = p^i_0 + n\hbar k^i_0 - \hbar k^i_1$.

Let's consider the reasonings from [3] attentively. Denoting $\delta p = p_1 - p_0$ and assuming that $p^2 = (p\cdot p) = m^2c^2$ one may conclude that $(\delta p\cdot p) = 0$. The latter would be true in the case

$$\delta p^i = \frac{(k_1\cdot p_0)}{(k_0\cdot p_0)}\hbar k^i_0 - \hbar k^i_1 \tag{12}$$

which the author used as the main assumption. But this means $(k_1\cdot p_0) = n(k_0\cdot p_0)$ and unclear why such a "rule of selection" must be true taking into account that wavevector $\mathbf{k}_1$ of the emitted photon may have any direction. For example, if the electron moves to meet the wave and is emitting a photon to the direction $\mathbf{e}_x$, then $\mathcal{E}_0(n\omega_0 - \omega_1)/c + p_{0x}(n\omega_0 + \omega_1) = 0$. The latter is obviously impossible when $\mathcal{E}_0/c \approx p_{0x}$.

Therefore equation (12) and thus (3) [3] are erroneous but this was not sole a mistake. In [3] the 2nd equation (6) with $f^i_L = f^i_{L0}$ was obtained by averaging the 1st equation (4) [3] in the co-moving frame of reference (MCLF). But in such a way one could only obtain that

$$\frac{dx^i_{rad}}{d\tau} = -\frac{f^i_{L0}}{f^2_{L0}}\int\frac{(p_0\cdot\hbar k_1)}{m\Delta\tau}\frac{\partial W}{\partial\omega_1}d\omega_1 = \frac{\tau_0}{mI_E}f^i_{L0}\frac{1}{\Delta\tau}\int_0^\infty \hbar\omega_1\frac{\partial W}{\partial\omega_1}d\omega_1 = \frac{\tau_0}{m}\frac{\widetilde{I}}{I_E}f^i_{L0}$$

In [3] the expression of $\widetilde{I}$ was arbitrarily replaced by the following one:

$$I = \int_{\omega_{min}}^\infty \hbar\omega_1\frac{dW}{d\omega_1 d\tau}d\omega_1 \tag{13}$$

The value of $\Delta\tau$ was neither defined nor explained. Such a heuristic reasoning may not be accepted as a derivation of the 2nd equation (6), even if one would assume that (12) is true.

Formula (13) looks strange because the integrand contains infinitely large factor $dW/d\tau d\omega_1$. Such mathematically abnormal expressions, where one d above and several d below, play an important part in [5]. The authors used them meaning the physical interpretations which are analogous to the following one. Namely, $W/d\tau$ denotes the probability of a photon emission per a unit of time. Herewith the symbol $d/d\tau$ is used as a differential operator. Such kind of tricks are discussed in §5.

In addition, the author postulated the 1st equation (6) referring to §73 [14]. But equation (73,3) [14] implies that $\dot{p}_{rad} = I_E/(mc^2) \cdot p^i$ where $p^i = m\dot{x}^i$ and I_E is the intensity of electro-dipole radiation. In [3] the latter was arbitrarily substituted with the fuzzy quantity I. Thus, any physical derivations of the 1st and 2nd equations (6) as well as (7) = (5) [3] are absent. In fact some equations, including erroneous, were arbitrarily adapted to the desired result.

Consider an analytic SEQ solution that is presented in §11 [2]. An electron is considered in homogeneous electrostatic field $\mathbf{E}$. A solution of system (8) is given by the formulas:

$$\begin{gathered}
p_x(\tau) = mc\sinh(\omega\tau), \quad p_{x,rad} = mc\omega\tau_0(\cosh(\omega\tau) - 1) \\
\mathcal{E} = mc^2\cosh(\omega\tau), \quad \mathcal{E}_{rad} = mc^2\omega\tau_0\sinh(\omega\tau) \\
x = c\omega^{-1}(\cosh(\omega\tau) - 1) + c\tau_0\sinh(\omega\tau) \\
t = \omega^{-1}\sinh(\omega\tau) + \tau_0(\cosh(\omega\tau) - 1), \qquad \omega = eE/(mc)
\end{gathered}$$

If $\tau_0 = 0$, these equations are reduced to the formulas in §20 [14]. However, here the parameter τ does not coincide with the proper time because $\dot{x}^i\dot{x}_i = c^2\dot{t}^2 - \dot{x}^2 = c^2(1 - \omega^2\tau_0^2) \neq c^2$. This means that $d\tau \neq \sqrt{1 - v^2/c^2} \cdot dt$. Therefore the dependencies $p_x = p_x(\tau)$, $\mathcal{E} = \mathcal{E}(\tau)$, $x = c\omega^{-1}(\cosh(\omega\tau) - 1)$, $t = \omega^{-1}\sinh(\omega\tau)$ don't represent the law of motion without accounting the Lorentz damping. In addition to distorting the proper time τ, another feature of Sokolov's model is the last formula (6) for the Lorentz force. It differs from the usual one and in this (1-dimensional) case the difference looks as:

$$\frac{e}{mc}F^{ij}p_j = \begin{bmatrix} f_L^1 \\ f_L^2 \end{bmatrix} = -eE\begin{bmatrix} \sinh(\omega\tau) \\ \cosh(\omega\tau) \end{bmatrix}, \quad f_L^2 \cdot \frac{d\tau}{dt} = -\frac{eE\cosh(\omega\tau)}{\cosh(\omega\tau) + \omega\tau_0\sinh(\omega\tau)}$$

$$\frac{e}{c}F^{ij}\dot{x}_j = \begin{bmatrix} f^1 \\ f^2 \end{bmatrix} = -eE\begin{bmatrix} \sinh(\omega\tau) \\ \cosh(\omega\tau) \end{bmatrix} - \frac{(eE)^2\tau_0}{mc}\begin{bmatrix} \cosh(\omega\tau) \\ \sinh(\omega\tau) \end{bmatrix}, \quad f^2 \cdot \frac{d\tau}{dt} = -Ee$$

where $i, j = 1, 2$. The error $|f^i - f_L^i|$ has the order of magnitude of the radiation reaction force because $\dot{p}_{x,rad} = (eE)^2\tau_0/(mc) \cdot \sinh(\omega\tau)$. It is also seen that eE value of the 3-force of Lorentz is not obtained from f_L^2 (don't confuse here the component f_L^2 with the sum $f_L^i f_{L,i}$). Thus, the distortion of the notion of Lorentz force in Sokolov's model is quite essential.

§ 3. Energy-momentum conservation

At the beginning of §9 [2] the author asserts that "*the difference from LAD is very small*" referring to the fact that all the different terms contain small factor τ_0. This argument doesn't hold water because τ_0 cannot be considered small without neglecting the effect under consideration. Obviously, the relative difference between solutions of (4) and (8) has the order of magnitude of the relative error that arises from neglecting the radiation reaction ($\sim \tau_0$). In fact, systems (4) and (8) are totally different. This is clearly seen when comparing (1) and (10).

As it is shown in §2, LAD was neither "modified" nor "renormalized" but simply distorted. The author asserts that his system (8) is more rigorous compared to (4). These assertions are based *solely* on the fact that, in contradistinction to (8), condition (P) is violated along solutions of (4). For any solution of LAD we have

$$p^2 = m^2(c^2 + \tau_0^2 \ddot{x}^2) < m^2c^2 \quad \text{if} \quad \ddot{x}^2 \neq 0. \tag{14}$$

The relative value ε of deflection (14) has the order $\sim \tau_0^2\ddot{x}^2/c^2$. Estimating the radiation reaction force as $dp^i_{rad}/dt \sim 10^4$ results in $\dot{p}^i_{rad}\gamma^{-1} = -m\tau_0\ddot{x}^2\dot{x}^i\gamma^{-1}/c^2 \sim 10^4$ dynes and, hence, $\varepsilon \sim 0.1\%$. Here $\gamma^{-1} = \sqrt{1 - v^2/c^2}$. In §10 [2] the applicability criteria of a classical approach is given:

$$\frac{\hbar^2 |f_L^2|}{m^2c^2} \ll m^2c^4 \tag{15}$$

Consider the case $|\mathbf{E}| = |\mathbf{H}| = E$ and $(\mathbf{E}, \mathbf{H}) = 0$ which is adequate to the problem of an electron in a strong laser field. We choose the coordinate axes such that $E_y = E_z = H_x = H_z = 0$ and $E_x = H_y = E$ in a fixed moment. Then

$$f_L^2 = f_L^i f_{L,i} = e^2/(mc)^2 \cdot F^{ij}F_{ik}p_jp^k = -e^2/(mc)^2 \cdot E^2\Big(\frac{\mathcal{E}}{c} - p_z\Big)^2 \tag{16}$$

Assuming $\gamma \sim 1$ the condition $|eE| \ll 10^4$ follows from (15) and (16). This estimate stays true when $\gamma \gg 1$. The giant force $\sim 10^4$ dynes would effect to an electron in a field with the intensity $\sim 10^{14}$ which is 3 - 4 orders greater than impulse lasers are able to generate today. Thus, the estimate $\varepsilon \ll 0.1\%$ looks realistic and the problem with (14) is not so dramatic as one might think while reading [2] and other articles by I.V. Sokolov.

The problem of rest mass, that is described in §7 [2], in fact is paltry. Let's estimate numerically the deflection of the rest mass of a motionless electron that has been accelerated by a force ($\mathbf{p} = 0$). If for such an electron we have $\mathcal{E} = (1-\delta)mc^2$ with $\delta \approx 0$, then $|\ddot{\mathbf{r}}| = \sqrt{2\delta} \cdot c/\tau_0$. According to (15) the Lorentz force $eE \ll 10^4$ dynes, hence $\delta \ll 0.001\%$.

Hence, shortcoming (14) represents a small fee for the fact that self-effect of a point charge is impossible to precisely describe in principle. As for the runaway solution of LAD mentioned in [2], which Dirac had considered in [1], this pathology is excluded by the physically natural condition of zero acceleration at $\tau = \pm\infty$. In general, the existence of non physical solutions of physical equations is neither new nor rare fact. Moreover, most of the solutions of any differential equations satisfy non physical initial or boundary conditions.

Across many articles starting from [2] the author of SEQ insistently convinces that his equations, in contradistinction to LAD, exactly fit to the energy-momentum conservation laws. In the following excerpt from [8] the author also criticizes LL for violation conservation laws. "*How theoretically important is the distinction between the two approaches? We discussed this issue in [3, 19] and noted that the LL equation conserves neither the generalized momentum of electron nor the total energy-momentum of the system consisting of an emitting electron, the external field and the radiation.*"

But (P) condition in itself implies no conservation. This can be trivially seen while considering a non-conservative mechanical system where some (not Lorentz) damping is not accounted in the equations of dynamics. The author of SEQ has presented neither rigorous proofs nor numerical estimates that in his case these laws hold, if not to account vague and puzzling reasonings like the following in §11 [2]. "*By choosing a particular gauge (pure scalar or pure vector potential), the conservation not only of energy, but also of generalized momentum can easily be ensured.*"

What is the point in varying the gauge of the trivial field which is considered at the end of §2 ? Since there takes place $p_x + p_{x,rad} = Eet$ and $\mathcal{E} - \mathcal{E}_0 + \mathcal{E}_{rad} = Eex$, there arises impression that the energy-momentum conservation law holds for this analytical solution. But here we have the impulse and the work of not the force which I.V. Sokolov refers to as the Lorentz force and denotes $\mathbf{f}_L$. Hence, the energy-momentum conservation does not hold in the framework of SEQ model.

The 1-st equation (2), which is valid for any expressions of $x^i_{rad}(t)$ and $p^i_{rad}(t)$, implies that

$$\frac{d(p^i + p^i_{rad})}{dt} = \frac{e}{c} F^{ij} \frac{dx_j}{dt}$$

From here it follows that for any t we have:

$$\mathbf{p} + \mathbf{p}_{rad} = \int_0^t \mathbf{f}(t')dt', \qquad \mathcal{E} - \mathcal{E}_0 + \mathcal{E}_{rad} = e \int_0^t \mathbf{E}(t') \cdot \mathbf{v}(t')dt'$$

where $\mathbf{f}$ is the 3-force of Lorentz. These equations express the energy-momentum conservation regardless of how the radiation losses p^i_{rad} have been modelled. Hence, LAD was unfoundedly charged with violation of conservation laws. Note that, in contradistinction to SEQ, true expression (0) for the Lorentz force is used in LAD and LL.

§ 4. Superluminal paradoxes

The following remark in §3.4 [13] relating to SEQ is worth to be mentioned: "*Then the notion of proper time breaks down, and we find that a massive particle must move at the speed of light (or faster!).*" The grounds for this assertion are considered in this section. They flow out from the following inequality which may be strict:

$$\dot{x}^2 = \dot{x}^i \dot{x}_i = c^2 + \frac{\tau_0^2}{m^2} f_L^2 \leq c^2 \tag{17}$$

The effects of the violation (V) are not limited to the fact that, as written in §10 [2], an electron does not move in the direction of its momentum. The latter also takes place in the case of LAD.

According to (15) the relative value ε' of deflection (17) has the order of $\tau_0^2 |f_L^2|/(m^2 c^2) \sim 10^{-13} \cdot (eE\gamma)^2$. Since $|eE| \ll 10^4$ then $\varepsilon' \ll 0.001\gamma^2$ %. Obviously that in ultra-relativistic case there may be $\varepsilon' \gg \varepsilon$ (see §3). Such a balancing act on the "relativistic edge" looks more dangerous than deflection (14) because of the infinitely increasing energy-momentum near the singularity $v = c$. In §10 [2] the author asserts that superluminal paradoxes in any case are excluded in his model. This opinion is based on the inequality

$$c^2(1 - \alpha^2) < \dot{x}^2 \leq c^2$$

where $\alpha \approx 1/137$ is the fine structure constant. Note than since $\dot{x}^i \dot{x}_i = (dt/d\tau)^2 \cdot (c^2 - v^2)$ then $\dot{x}^2 > 0$ is equivalent to $v < c$. In the case of SEQ the inequality $\dot{x}^2 > c^2(1 - \alpha^2)$ directly follows from (15) and (17) under the condition $f_L^2 \leq 0$ which is valid though not proved by the author.

Thus, condition (15) is the ground for the assertions that in the framework of Sokolov's model inequality $v < c$ is true for all t. Let's note that

$$\dot{x}^2 > c^2(1 - \alpha^2) \quad \Rightarrow \quad v < c \cdot \sqrt{1 - \frac{1 - \alpha^2}{\widetilde{\gamma}^2}} \quad \text{with} \quad \widetilde{\gamma} = \frac{dt}{d\tau} \neq \gamma$$

and the right-hand part tends to 1 as $v \to c$. This means the following. The closer the electron velocity to the speed of light the greater this velocity can be without violating the limit of classical approach applicability. The absurdity of this interpretation is a convincing evidence of the fact that SEQ is physically unfounded. Apparently, just this strange property became the ground for the myth that SEQ may be taken as a base for modelling QED-regimes.

Criterion (15) represents a Lorentz-invariant form of the condition $|\mathbf{f}| \ll 10^4$. The latter means that the distance traveled when accelerating up to the energy $\mathcal{E} \sim \mathcal{E}_0$ is many orders greater than Compton wavelength of the electron. In the case of laser field considered in §3 this results in $eE\gamma \ll 10^4$. But when applying (15) when $\gamma \gg 1$ there may be obtained questionable results. For example, in a laser field with the intensity $E \sim 10^9$ which corresponds to the energy flow $\sim 10^{20}$ W/cm^2 we would have $\gamma \ll 10^4$ and $\mathcal{E} \ll 10$ Gev. Thus, an electron with the energy ~ 1 Gev in such a field cannot be reliably modelled in a classical way, even if only the translational motion is considered.

Even more questionable conclusion from (15) one obtains for a field $\sim 10^{24}$ W/cm^2 which is envisioned in the nearest future [9]. Here only an electron with $\mathcal{E} \ll 100$ Mev can be classically described. Apparently, one should highly carefully use criterion (15) in ultra-relativistic case. In addition, it is not compatible with the claim that SEQ is well adequate to the case of super strong laser fields.

It is important to also remark that from (15) and (17) follows the more strong inequality $c^2(1-\alpha^2) \ll \dot{x}^2$, which isn't compatible with $\dot{x}^2 \leq c^2$. This is another evidence of physical absurdity of SEQ. Easy to verify that if τ parameter represents the proper time, inequality $\dot{x}^2 > c^2(1-\alpha^2)$ is equivalent to $v < c$ for all $\alpha \neq 0$. Thus, the author of SEQ in fact postulated that $v < c$ refusing to consider solutions of SEQ violating this fundamental limit. Nevertheless, superluminal paradoxes arise in the framework of his model.

Let's return to an electron in the plane wave considered in §3. Note that in the framework of Sokolov's model the equations $p^1 = mc\gamma$ and $(p^2, p^3, p^4) = m\mathbf{v}\gamma$ are invalid, otherwise we would have equation (V). Therefore now we are considering γ as simply such a factor that $\mathcal{E} = mc^2\gamma$.

Let the intensity $E = 2 \cdot 10^{11}$. That corresponds to the energy flow $\sim 10^{24}$ W/cm^2 in a laser pulse. Assume that $p_z \sim mc$. Then, according to (16) and (17), we have $\dot{x}^i\dot{x}_i = 0$ for some value $\gamma^2 \approx 2.1 \cdot 10^{11}$. Hence, an electron with energy $\mathcal{E} \approx 230$ Gev under the effect of this wave is able to achieve the speed of light. Such a situation is realizable in Bevatron. If the y-axis is directed along the axis of accelerator and the laser pulse is orthogonal to it, then one may obtain an electron which is moving faster than light! In standard physics the equation $\gamma^2 = 2.1 \cdot 10^{11}$ corresponds to the velocity $v \approx 0.9999999999977c$.

To exclude superluminal paradoxes from SEQ model the following assertion is utilized. The presented above example violates the Schwinger limit due to the Doppler effect in the co-moving frame of reference. But this argument contradicts to the declared applicability domain of SEQ. Indeed, in [4] the author writes the following. "*A covariant condition for the radiation reaction to be significant is as follows: (1)* (formula). *A high value of the integral in (1) may be reached, in principle, at the cost of higher intensity only,* $\sim 10^{25}$ W/cm^2. *In the course of the ELI project a laser is expected ... so the the radiation effects will be dominant. ... These equations* (SEQ) *may be applied to plasma electrons in order to simulate laser-plasma interactions at high laser field intensity.*

SEQ model is declared to be a part of research related to lasers with the extremal density of energy. In conclusion to [4]: "*This research is in the main stream of the ELI and HiPER* *projects"*. Thus, only in the case of energy flows $\sim 10^{25}\ W/cm^2$ and higher the radiation reaction becomes significant and SEQ, according to the author, is applicable for modelling the electron dynamics in such fields. In the example of superluminal motion we have $\sim 10^{24}\ W/cm^2$.

There is another argument in favor of SEQ. Namely, in the case of a QED - strong field this model should be used with $I = I_{QED}$ instead of $I = I_E$. Then, using the 2nd equation (6):

$$\dot{x}^2 = \dot{x}^i \dot{x}_i = c^2 + \frac{\tau_0^2}{m^2} \frac{I_{QED}^2}{I_E^2} f_L^2$$

According to [5] $I_{QED}/I_{cl} \sim \chi^{-4/3}$ (page 6). Hence, the superluminal paradoxes are excluded by the inequality $I = I_{QED} < I_E = I_{cl}$ when $\chi \gg 1$. Here the value $\dot{x}^2$ becomes closer to c^2 than in (17) for the same field and, hence, the inequality $c^2 - v^2 > 0$ is guaranteed.

Clarification: parameter χ estimates the ratio E_0/E_S where E_0 is the field intensity in the co-moving frame of reference (CMF) and E_S is the Schwinger limit (4) [5]. If $\chi \ll 1$ or $\chi \gg 1$, then QED - effects are inconspicuous or prevailing respectively.

This argument is fictional because the quantity I_{QED}, which is formally introduced in (28), in fact is meaningless (see §5). Even if to believe in the existence of this quantity, assigning some small value for the symbol $d\tau$, then I_{QED} becomes a fuzzy value defined up to proportionality (for various $d\tau$). Hence, the ratio I_{QED}/I_E may not be used when estimating the velocity of electron.

In addition, the smooth model, which is implied when calculating the electron velocity as $\dot{x}^i$, contradicts to the attempt of adapting SEQ equations to the discrete paradigm of quantum mechanics. At page 7 [5] the photon emission is depicted in the following way. "*The electron motion in the strong field may be thought of as the sequence of short intervals. Within each of these intervals the electron follows a piece of a classical trajectory ... The transition from one piece of the classical trajectory to another, or, the same, from one electron state to another occurs in a probabilistic manner.*"

According to this picture when a photon is being emitted, i.e., when the quantity I_{QED} becomes necessary the expression $\dot{x}^i$ is losing a sense. The matter is that the ends of time interval $\Delta\tau \to 0$ may relate to different smooth pieces of world lines. If the time moment $\widetilde{\tau}$ separates two such pieces, the limit $\lim_{\Delta\tau\to 0} \Delta x^i/\Delta\tau$ does not exist and $\dot{x}^i(\tau - 0) \neq \dot{x}^i(\tau + 0)$.

Summarizing §§3, 4: the assertions that system (8) is more rigorous compared to (4) are totally unfounded. This pretension looks strange taking into account that (8) was obtained solely from (P) condition imposed onto system (2) (see §2). The approach of Dirac was incomparably more rigorous and profound. It is based on the assumption that the flow of energy-momentum across the boundary of any infinitely thin 4-dimensional tube covering the world-line of an electron is determined by only the energy-momentum values at the ends of the tube [1]. This idea contains vastly more physics than the elementary condition (P).

§ 5. Quantum mechanical applications

In [9] SEQ is presented as the only recommended method for describing the dynamics of a radiating electron (§1.1.3 [9]). The following assertion from §1.3 [9] is worth to be mentioned: "*Particularly, emission of softer γ photons may even be described within the radiation force approximation, which is traditionally used to account for the radiation back reaction (see Refs. [32,34-36,64])*". Quite surprising is that Sokolov's approach, presented in 2009 [2], has already become traditional.

In §1.1.4 [9] equations (8) in the 3-dimensional form (21 - 23) [2] were used for describing the photon emission by a hydrogen-like atom. The was taken as a basis the difference equation:

$$\Delta(\mathcal{E} + e\varphi) = \omega \Delta M \quad (= -\hbar\omega) \tag{18}$$

which corresponds to a photon emission. Here the classical expression $\mathcal{E} + e\varphi$ for the energy of a hydrogen-like atom is used for the value $-R_y/n^2$ prescribed by the early theory of Bohr. In accordance to §1.1.4 [9] there take place the following equations:

$$\frac{d\mathbf{x}}{dt} = \mathbf{u} + \bar{\mathbf{u}} \qquad \bar{\mathbf{u}} = \frac{d\mathbf{x}_{rad}}{dt} = \frac{\tau_0}{m_e}\mathbf{f}_{Le} \qquad \frac{d\mathbf{p}}{dt} = \mathbf{f}_{Le} - \frac{d\mathbf{p}_{rad}}{dt}$$

$$\frac{d\mathbf{p}_{rad}}{dt} = \frac{\mathcal{E}^2(\bar{\mathbf{u}} \cdot \mathbf{f}_{Le})}{m_e^2 c^6}\mathbf{u} \qquad \frac{d\mathcal{E}}{dt} = (\bar{\mathbf{u}} \cdot \mathbf{f}_{Le}) - \frac{d\mathcal{E}_{rad}}{dt} \qquad \frac{d\mathcal{E}_{rad}}{dt} = \frac{\mathcal{E}^2(\bar{\mathbf{u}} \cdot \mathbf{f}_{Le})}{m_e^2 c^4} \tag{19}$$

Here $\mathbf{p} = \gamma m_e \mathbf{u}$ and $\mathbf{u} = [\vec{\omega} \times \mathbf{r}]$ where $\vec{\omega}$ is the electron angular velocity (in the classical sense), $\mathbf{f}_{Le} = [\vec{\omega} \times \mathbf{p}]/\gamma$ and $f_{Le} = Ze^2/r^2$.

According to this naive model the process looks as follows. Once a uniformly rotating electron started to emit a photon, its velocity instantly acquired the orthogonal component $\bar{\mathbf{u}}$ directed toward the nucleus. After the photon emission was completed, the electron instantly rotated and further moved along a circular orbit with a smaller radius. Although smooth equations (19) are not suitable for such a discontinuous model, the author decidedly replaces derivatives with the ratios of finite differences and back. The following relations were obtained:

$$\frac{d\mathbf{M}}{dt} \approx \left(\frac{\Delta \mathbf{M}}{\Delta t}\right)_{rad} = \left[\left(\frac{\Delta \mathbf{x}}{\Delta t}\right)_{rad} \times \mathbf{p}\right] + \left[\mathbf{r} \times \left(\frac{\Delta \mathbf{x}}{\Delta t}\right)_{rad}\right] \approx$$

$$\approx [\bar{\mathbf{u}} \times \mathbf{p}] - [\mathbf{r} \times \mathbf{u}]\frac{\gamma^2(\bar{\mathbf{u}} \cdot \mathbf{f}_{Le})}{c^2} = -\frac{\gamma^2(\bar{\mathbf{u}} \cdot \mathbf{f}_{Le})}{\omega^2}\vec{\omega} \quad \Rightarrow \quad \frac{d\mathbf{M}}{dt} \approx -\frac{\gamma^2(\bar{\mathbf{u}} \cdot \mathbf{f}_{Le})}{\omega^2}\vec{\omega} \tag{20}$$

$$\frac{d(\mathcal{E} + e\varphi)}{dt} = \frac{d\mathcal{E}}{dt} + \nabla(e\varphi) \cdot \bar{\mathbf{u}} = \frac{d\mathcal{E}}{dt} + (-\mathbf{f}_{Le}) \cdot \bar{\mathbf{u}} = -\gamma^2(\bar{\mathbf{u}} \cdot \mathbf{f}_{Le}) \approx \omega\frac{dM}{dt} \approx \frac{d(\mathcal{E} + e\varphi)}{dt} \tag{21}$$

The similarity between (18) and (21), according to the author, is a profound physical argument in favor of SEQ. But this analogy is highly superficial because ω for quasi-rotation of the electron and ω for the emitted photon are fundamentally different quantities (for the hydrogen atom they may differ from each other by one order of magnitude).

Moreover, the strict equality in the 2nd string of (20) contains a mistake. Indeed, after substituting the value $\gamma m_e \mathbf{u}$ in place of $\mathbf{p}$ this equality will cease to be valid when $\gamma \neq 1$. But the assumption $\gamma \approx 1$ does not correspond to the declared applicability domain for SEQ.

In §1.1.4 [9] equations (19) were used with the assumption that functions $\mathbf{x}_{rad}(t)$ and $-\mathbf{p}_{rad}(t)$ are amendments to the "unperturbed motion" $\mathbf{x}(t)$, $\mathbf{p}(t)$ which represents a uniform rotation of the electron around the nucleus. Using this naive approach exactly the same result can be obtained by means of LAD. Indeed, if $p^i = m\dot{x}^i$ and x^i is a solution of (0), then (4) coincides with (8). Repeating the calculations from §1.1.4 [9] under the same assumptions but using (4) instead of (8) one could obtain:

$$\frac{d\mathbf{p}_{rad}}{dt} = \frac{\tau_0 \gamma^2 |\mathbf{f}_{Le}|^2}{mc^2}\mathbf{u} \qquad \frac{d\mathcal{E}_{rad}}{dt} = \frac{\tau_0 \gamma^2 |\mathbf{f}_{Le}|^2}{m} \qquad \frac{d\mathbf{x}_{rad}}{dt} = \frac{\tau_0}{m}\mathbf{f}_{Le} \qquad \frac{d\gamma}{dt} = 0 \tag{22}$$

As easy to verify, equations (22) coincide with (19). The last approximation in (21) is also valid if $\gamma \approx 1$ or if the error $\mathbf{p} = m_e \mathbf{u}$ was made in (20).

Thus, §1.1.4 [9] bases itself on mathematical and physical mistakes. Amongst them is the last equation (22). It is obviously incompatible with this model which, in itself, looks too naive and far-fetched. Without regards to all these problems, the model obtained from SEQ may also be obtained from LAD.

The Sokolov's model is promoted as promising with regard to applications in QED - regimes, when an electron moves in such a strong field that QED effects are essential. Article [5] creates the impression that this value judgment is well founded. According to the abstract: "*The radiation force experienced by an electron is for the first time derived from the QED principles and its applicability range is extended toward the QED-strong fields.*"

The authors of [5] used the units system where $\hbar = m_e = c = 1$, mixing it with SGS based designations. Only SGS is utilized in what follows. Equations (40),(41) [5] are directly related to SEQ. Equation (41) [5] coincides with (7) and its last two terms represent the radiation reaction. The latter is formally identical to (40) [5]. This fact is the main ground for the allegations about QED applicability of SEQ model. Below it is shown that the coincidence was derived out of mistakes and inaccurate reasonings which were tailored to the desired result.

An electron with the energy-momentum p under the effect of a plane wave with the 4-wavevector $k = \omega/c \cdot (1, -\mathbf{e}_x)$ is considered in [5]. Let's analyze the final part of the article [5] starting from (39) [5]. It was obtained from the equation

$$\frac{dp_{rad}}{d\tau} = \hbar \int k' \frac{dW}{d\tau d^3\mathbf{k}'} d^3\mathbf{k}' \tag{23}$$

where $\mathbf{k}'$ is the 3-wavevector of an emitted photon and $dW/d\tau d^3\mathbf{k}'$ is the probability of this emission per a unit of time and a unit of volume in the space of wavevectors. Equation (23) looks quite reasonable if not to pay attention to a hidden trick which is discussed below. The upper string in (39) [5]

$$\frac{dp_{rad}}{d\tau} = \frac{\hbar}{m} \int k' \frac{(k \cdot p) dW}{d(k \cdot k') d^2\mathbf{k}'_\perp d\xi} d(k \cdot k') d^2\mathbf{k}'_\perp \tag{24}$$

directly follows from (23) using the formula at page 9

$$d^3\mathbf{k}' = \frac{\omega' d^2\mathbf{k}'_\perp d(k \cdot k')}{c(k \cdot k')} \tag{25}$$

and the identity (9) [5] for the dimensionless variable $\xi = (k \cdot x)$ defined along the world line:

$$\frac{d}{d\tau} = \frac{(k \cdot p)}{m} \frac{d}{d\xi} \tag{26}$$

Vector $\mathbf{k}'_\perp$ is orthogonal to $\mathbf{e}_x$. The right-hand part of (25) is equal to $d^2\mathbf{k}'_\perp \omega' dn'_x/(c(1+n'_x))$, where n'_x is the x - projection of the vector $\mathbf{n}' = \mathbf{k}'/|\mathbf{k}'|$. From here it is clear that (25) is true only when $n'_x = 0$, i.e., if the following takes place:

$$(k \cdot k') = \frac{\omega}{c} \frac{\omega'}{c} \tag{27}$$

Since (27) does not hold for almost all the photons, formulas (25) and (24) = (39) [5] are invalid.

In (23) and (24) is used a trick analogous to that we have seen in (13). These formulas are meaningless due to the integrands containing infinitely large factors, for example $dW/d\tau d^3\mathbf{k}'$. The trick consists of the interpretation that in $dW/d\tau d^3\mathbf{k}'$ the symbols $d\tau$ and $d^3\mathbf{k}' = dk'_x dk'_y dk'_z$ don't imply the derivatives with respect to τ, k'_x, k'_y, k'_z but play the role explained below (23).

Such an interpretation looks physically evident, but it has a mathematical consequence. Namely, for some function $\mathcal{W}$ there must be:

$$\frac{dW}{d\tau d^3\mathbf{k}'} = \frac{\partial^4 \mathcal{W}}{\partial\tau \partial k'_x \partial k'_y \partial k'_z}$$

Impossible to physically interpret the "probability" $\mathcal{W}$. More dramatic is that from the integral (23) it follows that $\mathbf{p}_{rad} = const$. Indeed, integrating (23) by parts when $(k')^i \in \{k'_x, k'_y, k'_z\}$, one obtains zero due to the evident fact that $\mathcal{W} = 0$ if at least one from the cases $k'_x = \pm\infty$, $k'_y = \pm\infty$, $k'_z = \pm\infty$ holds (in the limit). The same follows from the equation (24).

The lower string in (39) [5] was obtained using the parameter $r_0 = \hbar(k \cdot k')/(\chi(k \cdot p))$ (page 11 [5]). Unclear why the author neglected the summand $kO((k \cdot p)^{-1})$ in the square brackets. In the system of units which is used in [5] this value might have the order of magnitude $p \sim 1$, if equation (39) [5] is considered in CMF at a fixed moment.

Thus, equation (39) [5] does not hold water. The assertion just above (39) [5], that it can be found from (30),(38) [5], looks highly questionable. Evidently that (38) [5] has no relation to the obtaining (39) [5]. To discloser the other mistakes, in what follows we assume that this equation is valid and the 2nd summand in the square brackets is equal to zero.

The following equations are given below (39) [5]:

$$\frac{dp_{rad}}{d\tau} = \frac{p}{mc^2} \int \frac{dI_{QED}}{dr_0} dr_0 \qquad I_{QED} = \hbar c^2 \int (k \cdot k') \frac{dW}{d\xi dr_0} dr_0 \tag{28}$$

Below equations (28) the author writes: "*the photon energy spectrum, dI_{QED}/dr_0, is described as a function only of the random scalar r_0, using only the parameter χ*". Translating into mathematical language one should conclude that $I_{QED} = I_{QED}(r_0)$ and equation

$$\frac{dp_{rad}}{d\tau} = \frac{I_{QED}}{mc^2} p \tag{29}$$

does not follow from the 1st equation (28). Indeed, the integral is equal to $I_{QED}(r_{0,max}) - I_{QED}(r_{0,min})$. It should be zero because $I_{QED}(r) = 0$ when $r_0 > r_{0,max}$ or $r_0 < r_{0,min}$. This confusion arises from the attempts to utilize "spectrums" while ignoring their mathematical nature (see above).

But there arises the impression that (29) results from (39) [5] and the 2nd equation (28). The latter was obtained from the 1st equation (6) and (13) assuming $I = I_{QED}$. But (13) and the 2nd equation (28) are different even in the formal sense. Indeed, from (26) and the 2nd equation (28) one may formally obtain:

$$I_{QED} = mc^2 \chi \, r_0 \frac{W}{d\tau}\bigg|_{r_{0,min}}^{r_{0,max}} - \hbar c^2 \int \frac{W}{d\xi} d(k \cdot k') = mc^2 \chi \, r_0 \frac{W}{d\tau}\bigg|_{r_{0,min}}^{r_{0,max}} - mc^2 \chi \int \frac{W}{d\tau} dr_0 \tag{30}$$

It is easy to analogously derive from (13):

$$I = \hbar \omega_1 \frac{W}{d\tau}\bigg|_{\omega_{1,min}}^{\omega_{1,max}} - \hbar \int \frac{W}{d\tau} d\omega_1 \tag{31}$$

The right-hand parts of (30) and (31) are formally equal only if (27) takes place. But the equality (27) is not true almost always. Hence, the 2nd equation (28) could not be formally obtained. Analyzing the other errors let's ignore the fact that (29) is false.

According to [5], while emitting a photon the electron acquires from the field the energy-momentum:

$$dp_F^i = \frac{(k' \cdot p)}{(k \cdot p) - \hbar (k \cdot k')} \hbar k^i \tag{32}$$

(page 12) [5]. Note, this equation contradicts to (3) [3] where $\hbar(k \cdot k')$ is absent.

When moving from (39) [5] to the equation (40) [5] the author neglected the value $\hbar(k \cdot k')$ in (32) (just above (40) [5]). Thus, equation (12) was obtained. Ignoring the fact that (12) is false (§3), this "approximation" is worth to be discussed as such. At page 12 it is characterized to be very convenient. But the convenience in obtaining the desired result is not a ground for recognizing the derivation correct. This rough approach means $\hbar(k\cdot k')/(k\cdot p)\approx 0$ and, hence, the energy-momentum of the emitted photon is negligible compared to that of the electron. This contradicts to the declared SEQ applicability near the QED - regimes. Besides, the parameter r_0 is considered to be negligible. This contradicts to the equations (28)

Then the author defined the radiation force as $(dp_F - dp_{rad})/d\tau$ (page 12), although this force must be equal to $(-dp_{rad})/d\tau$. Generally speaking, in the process of radiation the electron does not waste all the energy-momentum dp_F and thus is accelerating. Leaving these blunders let's follow [5] further.

Using the 2nd equation (28) with (26) and (27) one may formally obtain:

$$\int \frac{(p\cdot k')}{(p\cdot k)}\hbar k \frac{dW}{d\tau d(k\cdot k')}d(k\cdot k') = \frac{(p\cdot p)}{(p\cdot k)}k\cdot\frac{I_{QED}}{mc^2} \tag{33}$$

Here we have made what the author wrote about: "*The total radiation force may now be found by integrating dp_F over $d(k\cdot k')$*" (just above (40) [5]). Equation (40) [5] instantly follows from (29) and (33), but (29) is invalid and the equation (27) does not hold for almost all the wavevectors k' which the emitted photon may have.

Besides, the trick with using scalar r_0 in place of 4-vector k' while considering the spectrum $dW/d\xi dr_0$ (28) has the following background. The author assumed that the probability of emission to the direction k' depends only on the value $(k\cdot k')$. Easy to show that this assumption is unfounded. For example, let one photon be emitted at the angle $\pi/3$ to the direction $\mathbf{k}$ and another photon be directed orthogonal to $\mathbf{k}$. If their 4-wavevectors are k' and k'', then in the case $(k'\cdot k) = (k''\cdot k)$ we have $\omega' = 2\omega''$. According to [5] the probabilities of emission for these photon are equal. Evidently, this conclusion is baseless in the framework of quantum mechanics.

Moreover, what does express the ratio $d(p_f - p_i)/d\tau$ when an electron is emitting a photon and its energy-momentum is changing from p_i to p_f ? Here i and f denote "initial" and "final". Anyway, equation (40) [5] results in physically absurd conclusions. For example, if the electron moves to the direction $\mathbf{n}$ of propagating of the wave,

$$\frac{(p_i\cdot p_i)}{(k\cdot p_i)}k - p_i = \big(|\mathbf{p}|, \mathcal{E}\mathbf{n}/c\big) \qquad \frac{d(\mathcal{E}_f - \mathcal{E}_i)}{d\tau} = c|\mathbf{p}| \qquad \frac{d(\mathbf{p}_f - \mathbf{p}_i)}{d\tau} = \frac{\mathcal{E}}{c}\mathbf{n}$$

Ignoring the meaninglessness of the left-hand part of the last equation, its right-hand part means that the radiation force accelerates the electron regardless the direction a photon is emitting to! This absurdity clearly demonstrates the physical groundlessness of formula (40) [5].

In addition, the quantity I_{QED} in (41) [5] was not derived strictly but appointed as a value for the ambiguous quantity I (7). Thus, all the reasonings from [5] related to SEQ don't hold water. This confusion arose due to the attempt to describe quantum effects by means of a purely classical model (6).

§ 6. Conclusion

In the course of analysis presented in this article there were found no objective grounds for the assertions that SEQ is more rigorous than LAD and more suitable in the case of super-strong fields. This opinion was formed by the author himself by active advertising of the advantages of his equations as if they were reliably confirmed. But no experiments substantiating these pretensions were conducted. Moreover, neither rigorous proofs nor numerical evaluations were presented which would indicate that SEQ is at least equally adequate as LAD.

In fact, SEQ represents not a modification but a distortion of LAD. Acting in similar manner one may obtain infinitely many equations for the same physical problem, each of them neither experimentally confirmed nor rigorously derived from unambiguous assumptions. Two examples of this kind are presented in §2. One of them is similar to SEQ but looks more physically adequate (11).

The violation of condition $p^i p_i = m^2c^2$ in the framework of LAD is insignificant because the relative error $\varepsilon \ll 0.1\%$ (§3). Herewith Sokolov's model (SEQ) violates the identity $\dot{x}^i\dot{x}_i = c^2$, which may lead to dramatic consequences. Amongst them are superluminal paradoxes. An example of such a SEQ solution is given in §4. Contrary to what the author asserted in numerous articles, the energy-momentum conservation holds in the case of LAD and does not hold in the framework of SEQ (§ 3). Thus, there exist no advantages of SEQ compared to LAD.

The widespread opinion that SEQ model is especially appropriate for using in QED - regimes is also unfounded. Moreover, the corresponding results are very questionable due to numerous mistakes of different nature (§5).

Comparison of the articles [1] and [2] clearly shows that SEQ represents the result of purely mathematical speculations without physical grounds. Of course, this model has a right to be considered amongst other ones. But in present don't exist reasons for recognizing Sokolov's equations as a profound and promising approach having any advantages over LAD, contrary to what have been written in numerous articles.

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