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The angular momentum of an electromagnetic wave, the Sadovskiĭ effect, and the generation of magnetic fields in a plasma

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This paper discusses problems associated with the expressions for the angular momentum density and the flux density of the angular momentum of an electromagnetic wave within the framework of classical electrodynamics. We show that the formal application of the quantum relationships for a photon to the calculation of the flux density of the angular momentum yields false results. For a plane electromagnetic wave this quantity equals zero despite the opinion established in the literature. This conclusion does not contradict the possibilities of angular momentum transfer in the interaction of a circularly polarized wave with material objects. In particular, in the absorption of such a wave in a plasma, angular momentum is partly transferred to the electrons, and closed quasistationary electric currents arise. The process of generation of the magnetic field excited here possesses a number of distinctions from the inverse Faraday effect known in the literature.

1. INTRODUCTION

This methodological note is intended to clarify a firmly established misunderstanding concerning the expression for the angular momentum of a plane electromagnetic wave in classical electrodynamics. It turns out that the analysis of even the simplest effects of angular momentum transfer in the interaction of electromagnetic waves with matter requires correction of the views that have become established.

For example, in the review (Ref. 1), by analogy with the known quantum expressions for the energy of a photon ($\mathscr{E} = h\omega$) and for the projection of the angular momentum on the direction of propagation ($m = \pm h$ for a photon with right- or left-hand circular polarization), according to which

$$\mathscr{E}/\omega = \pm m, \qquad (1)$$

it is stated that the flux density of angular momentum in a circularly polarized plane electromagnetic wave is expressed in terms of the flux density of energy (the Umov–Poynting vector[2] S_α) by the formula

$$g_{xx} = \pm S_x/\omega\ ; \qquad (2)$$

Here g_{xx} is understood to be the flux density of the angular momentum in the x direction, and it is assumed that the wave $\sim \exp(-i\omega t + ikx)$ is propagating along the x axis of coordinates. The relationship (2) does not contain Planck's constant h, just like Eq. (1), and is assumed to be valid (although this has not been specially verified) also within the framework of classical electrodynamics. Poynting[3] first proposed as a hypothesis such a relation between the flux densities of energy and angular momentum starting from nonrigorous mechanical analogies.

As an experimental proof of the validity of Eq. (2) one usually cites the results of Refs. 4 and 5. They found the action of a torque on an object that absorbs circularly polarized electromagnetic radiation or on a $\lambda/4$ plate that changes the polarization of the radiation. This effect bears the name of A. I. Sadovskiĭ, who predicted it (1899) and offered a quantitative calculation.[6] Interestingly, Sadovskiĭ recalled also the earlier attempt of Riga (?) to bring into rotation "objects suspended by very fine quartz filaments using a beam of circularly polarized electromagnetic waves and using a beam of light rays of the same type."

The views that were presented on the flux density of angular momentum of an electromagnetic wave seem plausible, and they entered into Refs. 7 and 8. At the same time, the attempt to substantiate Eq. (2) in greater detail (hopeless from the outset, since this relationship is false and a plane wave has $g_{xx} = 0$) compels us to discuss the flux of photons passing through unit area—since we are discussing flux densities. But if a photon "climbed through" such an area, then the uncertainty of the momentum that arose here leads to disagreement between the direction of the wave vector of the photon and that of the normal to the area (the x axis). The projection of the angular momentum on the x axis becomes uncertain, since the expression $m = \pm h$ is valid only for the projection of the angular momentum on the direction of propagation. Therefore the unquestioned relationship (1) cannot be directly transferred to a quantity of the type of the density of angular momentum or flux density of angular momentum.

It is also impossible to prove Eq. (2) experimentally. To measure experimentally precisely the flux density of angular momentum in a plane wave, rather than the total flux of angular momentum in a beam of waves (which actually proves to be finite owing to edge effects), the dimensions of the object set into rotation must be small in comparison with the scale of the transverse distribution of the amplitude in the wave incident on the object. But here one cannot state that the angular momentum imparted to the object is localized in the plane wave incident on the object, since the conservation law is actually satisfied by the removal of an angular momentum opposite in sign in the wave diffracted by the object, which is not plane.

The aim of our note is not so much to refute Eq. (2) as to try to give correct classical analogs of the quantum relationship (1) and to trace more carefully the fulfillment of the laws of conservation of angular momentum in the Sadovskiĭ

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effect. Moreover, it is of interest to analyze the Sadovskiĭ effect in the absorption of a circularly polarized electromagnetic wave in a plasma. Here the angular momentum is partially transmitted to the low-inertia electrons and accelerates them more effectively than in the action of a wave on solid objects.

It turns out that in a plane electromagnetic wave $g_{xx} = 0$. A beam of electromagnetic waves of finite cross section Σ has a finite flux of angular momentum, with its magnitude associated with the energy flux by a relation like Eq. (1), which is the classical analog of the helicity of a photon. The flux density of angular momentum is localized at the edges of the beam, where the wave is not plane. Interestingly, the "helicity density," as would be appropriate to call the quantity $g_{\alpha\beta}S_\alpha S_\beta/S^2$, is identically zero for an arbitrary electromagnetic field.

It was shown that the manifestation of the Sadovskiĭ effect in the absorption of a circularly polarized wave in a plasma leads to the appearance of entrainment currents[9,10] of specific configuration and to generation of time-independent magnetic fields.

2. ANGULAR MOMENTUM OF AN ELECTROMAGNETIC FIELD

As is known, the angular momentum of an electromagnetic field is defined by the following integral[2,11,12] (r^i is the four-dimensional radius vector, and P^i is the 4-momentum:

$$M^{ik} = \int r^i dP^k - \int r^k dP^i = c^{-1}\int (r^i T^{kl} - r^k T^{il}) dS_l \qquad (3)$$

(notation as in Ref. 21). Here the tensor of the angular momentum density according to (3) is

$$m^{ikl} = c^{-1}(r^i T^{kl} - r^k T^{il}) ; \qquad (4)$$

Here T^{ik} is the symmetrized energy-momentum tensor for a free electromagnetic field:

$$T^{ik} = \begin{pmatrix} W & \frac{1}{c} S^\alpha \\ \frac{1}{c} S^\alpha & \sigma^{\alpha\beta} \end{pmatrix}, \qquad (5)$$

while W, S^α, and $\sigma^{\alpha\beta}$ are the energy density, the Poynting vector (flux density of energy and simultaneously the momentum density multiplied by c^2), and $\sigma^{\alpha\beta}$ is the Maxwell tensor of momentum flux:

$$W = \frac{1}{8\pi}(E^2 + B^2), \qquad (6)$$

$$S_\alpha = \frac{c}{4\pi}[\mathbf{EB}]_\alpha, \qquad (7)$$

$$\sigma_{\alpha\beta} = \frac{1}{4\pi}\left[\frac{1}{2}\delta_{\alpha\beta}(E^2 + B^2) - E_\alpha E_\beta - B_\alpha B_\beta\right], \qquad (8)$$

E and **B** are the electric and magnetic fields. For the three-dimensional quantities with Greek indices in (6)–(8) and later, we make no distinction between covariant and contravariant components.

The law of conservation of energy-momentum $\partial T^{ik}/\partial r^k = 0$ for the symmetrized tensor T^{ik} leads to an expression for the law of conservation of the angular momentum in the form of the equality to zero of the four-dimensional divergence:

$$\frac{\partial}{\partial t} m^{\alpha\beta 0} + \frac{\partial}{\partial r^\gamma} c m^{\alpha\beta\gamma} = 0 . \qquad (9)$$

After converting to dual quantities by multiplying by $e_{\alpha\beta\gamma}$ ($e_{\alpha\beta\gamma}$ is the three-dimensional asymmetric unit tensor), this yields

$$\frac{\partial}{\partial t}\left[\mathbf{r}\frac{\mathbf{S}}{c^2}\right]_\delta + \frac{\partial}{\partial r_\gamma}\left(e_{\delta\alpha\beta} r_\alpha \sigma_{\beta\gamma}\right) = 0 . \qquad (10)$$

Equation (10) relates the change in angular momentum density to the three-dimensional divergence of the flux density of angular momentum

$$g_{\delta\gamma} = e_{\delta\alpha\beta} r_\alpha \sigma_{\beta\gamma} = \frac{1}{4\pi}\left\{-[\mathbf{rE}]_\delta E_\gamma - [\mathbf{rB}]_\delta B_\gamma + \frac{E^2 + B^2}{2} e_{\delta\alpha\gamma} r_\alpha\right\}. \qquad (11)$$

Evidently the quantities $g_{\delta\gamma}$ have the physical meaning of the magnitude of the δth component of the angular momentum transferred per unit time through unit area perpendicular to the γth coordinate axis. The expression for $g_{\delta\gamma}$ contains three terms that describe the transport of angular momentum along the electric field, along the magnetic field, and the transport of the component of the angular momentum directed along $[\mathbf{rn}]$ along any vector $\mathbf{n}$.

3. ZERO "HELICITY DENSITY"

We note the interesting identity

$$g_{\delta\gamma} S_\delta S_\gamma = 0 , \qquad (12)$$

which follows from (11) and from the identities $S_\gamma E_\gamma = 0$, $S_\gamma B_\gamma = 0$, and $e_{\delta\alpha\gamma} S_\delta S_\gamma = 0$. It implies that no component of the angular momentum oriented along the Poynting vector can be transported along the Poynting vector. In particular, in a plane electromagnetic wave propagating along the x axis, the Poynting vector is directed along x, and hence $g_{xx} = 0$. In view of the latter condition, the relationship (2) is never fulfilled.

4. REMARK ON THE VARIATION OF THE ANGULAR MOMENTUM TENSOR IN A DISPLACEMENT TRANSFORMATION

Here the position of the point (coordinate origin) with respect to which the angular momentum is defined is not fixed in any way. Generally speaking, when the reference origin is displaced the angular momentum tensor is altered. However, defined directions exists such that the projection of the angular momentum on this direction remains invariant for any displacement of the coordinate system. For example, in mechanics the angular momentum of a material point upon displacement of the reference origin by the vector $\delta\mathbf{r}$ varies by $\delta\mathbf{M} = [\delta\mathbf{r}\mathbf{p}]$ ($\mathbf{p}$ is the momentum), but its projection on the direction of $\mathbf{p}$ remains invariant, since $\delta\mathbf{M}$ is perpendicular to $\mathbf{p}$.

Analogously in (12) and everywhere below, where the projections of angular momentum fluxes in individual directions are calculated, the results prove to be invariant upon a displacement transformation.

5. REMARK ON THE CANONICAL ANGULAR MOMENTUM

The derivation of the law of conservation of angular momentum by using the Noether theorem (see, e.g., Ref. 14) leads to the canonical expression for the angular momentum in the form of the sum of the orbital and spin moments:

$$m^{(c)}_{ijk} = m^{(o)}_{ijk} + S_{ijk}, \tag{13}$$

where

$$m^{(o)}_{ijk} = \frac{1}{c}(x_i T^{(c)}_{jk} - x_j T^{(c)}_{ik}),\quad S_{ijk} = \frac{1}{4\pi c}(-A_i F_{jk} + A_j F_{ik}), \tag{14}$$

A_i and F_{ij} are the 4-vector of the potential and the field tensor:

$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}, \tag{15}$$

while the canonical (asymmetric) energy-momentum tensor equals

$$T^{(c)}_{ij} = T_{ij} + \frac{1}{4\pi}\frac{\partial A_i}{\partial x_l}F_{lj}. \tag{16}$$

For a free field the difference between the canonical tensor of the angular momentum $m^{(c)}_{ijk}$ and the tensor m_{ijk} used in Ref. 2 and in the present paper is the four-dimensional divergence of the tensor B_{ijkl}, which is asymmetric in the last pair of indices, and adding it does not alter the law of conservation of angular momentum (9). Therefore for a free field the expressions (4) and (13) are equally valid. However, in analyzing the transfer of rotational motion from a field to charged particles, the tensors m_{ijk} and $m^{(c)}_{ijk}$ are no longer equally valid, since in the presence of charges their difference is expressed in terms of the current density j_k:

$$m^{(c)}_{ijk} - m_{ijk} = \frac{\partial B_{ijkl}}{\partial x_l} - \frac{1}{c^2}(x_i A_j - x_j A_i) j_k. \tag{17}$$

In view of Eq. (17) only the tensor m_{ijk} gives the angular momentum density of the field, which (the angular momentum) is conserved in the sum with the angular momentum of the mechanical momentum of the charged particles $\mathbf{M} = [\mathbf{rp}]$ (the conservation law is given in Ref. 13, and directly follows from the conservation law proved in Ref. 2 for the tensor T_{ij} in the sum with the energy-momentum tensor of the free particles, if one takes account of the symmetry of these two tensors). At the same time, the canonical tensor $m^{(c)}_{ijk}$ leads, as one can easily show by using (17), to the canonical angular momentum of the field, which is conserved in the sum with the angular momentum of the canonical momentum of the particles:

$$\mathbf{M}^{(c)} = \left[\mathbf{r}\left(\mathbf{p} + \frac{e\mathbf{A}}{c}\right)\right]; \tag{18}$$

Here e is the charge.

The quantity $\mathbf{M}^{(c)}$ generally does not coincide with $\mathbf{M}$. Therefore it is incorrect to identify the change in the canonical angular momentum of the field in the interaction with objects with the transfer of rotational angular momentum to the objects.

6. ON THE SADOVSKIĬ EFFECT IN THE INTERACTION OF AN ELECTROMAGNETIC WAVE WITH A DIPOLE

Let us trace the fulfillment of the law of conservation of angular momentum in the interaction of a plane electromagnetic wave with a time dependent dipole moment. We shall show that, although an angular momentum can be transferred to the dipole, nevertheless this angular momentum is not drawn from the plane wave incident on the dipole.

Let the plane wave having the electric field $\mathbf{E} = \mathbf{E}_0 \exp(-i\omega t + ikx)$ be incident on a dipole located at the coordinate origin, whose (electric) moment is perpendicular to the x axis and which depends on the time as $\mathbf{d} = \mathbf{d}_0 \exp(-i\omega t)$. The complex character of the quantities $\mathbf{e}_0$ and $\mathbf{d}_0$ allows for the possibility of elliptical polarization of the wave, and also rotation and/or oscillation of the dipole in the plane perpendicular to the x axis.

This problem is closely associated with the Sadovskiĭ effect, especially with an experiment of the type of Ref. 5, when waves are used in the UHF range, while the dimensions of the object to which the rotational angular momentum is transferred are smaller than the wavelength. Here a dipole moment of the object arises as a result of its polarization in the field of the wave. If the x axis coincides with one of the principal directions of the polarizability of the object, the vector $d_\beta = \alpha_{\beta\gamma} E_\gamma$ is perpendicular to the x axis ($\alpha_{\beta\gamma}$ is the polarizability tensor).

The field created by the electric dipole at the point $\mathbf{R}$ equals[13]

$$\mathbf{B}_\mathbf{d} = k^2 \frac{[\mathbf{Rd}_0]}{R^2}\left(1 - \frac{1}{ikR}\right)\exp(-i\omega t + ikR), \tag{19}$$

$$\mathbf{E}_\mathbf{d} = \left\{k^2 \frac{[[\mathbf{Rd}_0]\mathbf{R}]}{R^3} + \left(\frac{3\mathbf{R}(\mathbf{Rd}_0)}{R^2} - \mathbf{d}_0\right)\left(\frac{1}{R^3} - \frac{ik}{R^2}\right)\right\}\exp(-i\omega t + ikR). \tag{20}$$

Let us calculate the time-averaged flux of angular momentum G_{xx} through the plane $x = \pm L$, $kL \to \infty$. Here we assume that $|\mathbf{E}_0| \gg k^3|\mathbf{d}_0|$ (i.e., $k^3\alpha_{\beta\gamma} \ll 1$), and we neglect the terms quadratic in $\mathbf{B}_\mathbf{d}$ and $\mathbf{E}_\mathbf{d}$. We have

$$G_{xx} = \frac{1}{8\pi}\mathrm{Re}\int dy\,dz\left\{\frac{k^2 x}{R^3}[\mathbf{R}_\perp \mathbf{E}_0](\mathbf{R}_\perp \mathbf{d}_0^*) - \frac{k^2}{R^2}[\mathbf{R}_\perp \mathbf{d}_0^*](\mathbf{R}_\perp \mathbf{E}_0)\right\} \times \exp(-ikR + ikx), \tag{21}$$

the vector $\mathbf{R}_\perp$ in the plane $x = \mathrm{const}$ has y and z components; and the asterisk here and below denotes the complex conjugate.

The integrals of the type of (21) as $kL \to \infty$ are evaluated by the stationary-phase method.[15,16] Calculation of the contribution from the point $\mathbf{R}_\perp = 0$ at which $\partial R/\partial \mathbf{R}_\perp = 0$ yields

$$G_{xx} = \begin{cases} 0, & x = -L, \\ -\frac{1}{2}\mathrm{Re}\,[\mathbf{d}_0\mathbf{E}_0^*], & x = L, \end{cases} \tag{22}$$

as $kL \to \infty$.

Thus, in the incident wave a flux of angular momentum is absent, while in the wave that has undergone interaction with the dipole, the finite flux of angular momentum is induced $G_{xx} = -\mathrm{Re}\,[\mathbf{d}_0\mathbf{E}_0^*]/2$.

On the other hand, the time average of the change in the angular momentum of the dipole dM_x/dt, i.e., the moment of the forces acting on the dipole, equals

$$dM_x/dt = \mathrm{Re}\,[\mathbf{d}_0\mathbf{E}_0^*]/2\,, \qquad (23)$$

as is implied by time-averaging of the known expression for [**dE**]. Thus the law of conservation of angular momentum is fulfilled in the form $dM_x/dt + G_{xx}(x \to +\infty) = 0$.

We see that the Sadovskiĭ effect in the interaction of a wave with a dipole is reduced to the action on the dipole of the time-averaged torque of (23) and to inducing a flux of angular momentum G_{xx} of opposite sign in the wave diffracted by the dipole. Thus the experimental measurement of the quantity $dM_x dt$ can serve only to verify the unquestioned relationship (23)—but not as a proof of the existence of an angular momentum in the wave incident on the dipole.

The quantity in (23) differs from zero, e.g., for a linearly polarized wave in the case in which the two principal values of the polarizability tensor in the yz plane are not equal to one another. The torque here tends to rotate the dipole so that the axis having the maximum value of the polarizability is directed along the electric field. The case is also interesting in which the polarizability tensor in the yz plane has a positive imaginary component (absorption): $\alpha_{\beta\gamma} = (\mathrm{Re}\alpha + i\mathrm{Im}\alpha)\delta_{\beta\gamma}$, and the dipole interacts with a circularly polarized wave. Here the quantity dM_x/dt equals $|\mathbf{E}_0|^2\mathrm{Im}\alpha/2$ and is associated with the absorbed power $d\mathscr{E}/dt = \omega\mathrm{Im}\alpha|\mathbf{E}_0|^2/2$ by the relationship

$$\frac{dM_x}{dt} = \pm\,\omega^{-1}\frac{d\mathscr{E}}{dt}, \qquad (24)$$

which is the exact classical analog of the quantum relationship (1). Equation (24) can be interpreted as the manifestation of the quantum laws of conservation of energy and angular momentum in the absorption of a photon—but not as a proof of the existence of an angular momentum flux in a plane wave.

We emphasize that Sadovskiĭ himself considered the main result of his study[6] to be precisely the derivation of Eq. (23) for the case of monochromatic electromagnetic waves and its use for calculating the torque acting on various plates. Only one short remark was made on the angular momentum of the field (a correct one!—see below), which did not pertain to a plane wave.

7. THE FLUX OF ANGULAR MOMENTUM FROM A ROTATING DIPOLE

Although the effect of loss of angular momentum owing to radiation in the rotation of an electric dipole is well known in the literature (see, e.g., Ref. 13), for the sake of completeness we shall present here the expression for the flux of angular momentum through an infinitely remote sphere whose center coincides with the site of the dipole. According to (11) and (19) we have

$$\frac{dM_x}{dt} = \frac{k^3}{3}\,\mathrm{Im}\,[\mathbf{d}\mathbf{d}^*] = \pm\,\omega^{-1}\frac{d\mathscr{E}}{dt}. \qquad (25)$$

That is, Eq. (24) is valid both in emission and in absorption.

8. THE SADOVSKIĬ EFFECT WITH A PLATE OF FINITE AREA

Let the plane electromagnetic wave $\mathbf{E}_\omega = \mathbf{E}_{\omega 0}\exp(-i\omega t + ikx)$ be incident on a plane plate that is parallel to the yz plane and which has the finite area $\Sigma \gg k^{-2}$. If the permittivity of the material of the plate is anisotropic, then let the direction of the x axis coincide with one of the principal directions of the tensor $\varepsilon_{\alpha\beta}$. The quantities $\varepsilon_{\alpha\beta}$ are assumed to depend only on x, while diffraction in the volume of the plate is neglected.

Such a plate can acquire a moment of momentum owing to a change in the amplitude or polarization characteristics of the wave as the electromagnetic wave passes through the plate or is reflected. For a quantitative analysis of the effect of transfer of angular momentum to the plate from the field, we can calculate the angular momentum borne away by the reflected and transmitted waves.

The resulting electric field $\mathbf{E}$ is the sum of $\mathbf{E}_\omega$ and the field $\mathbf{E}_d$, which is created by the currents flowing in the plate. The vector of the dipole-moment density in the plate is perpendicular to the x axis owing to the assumptions that were made, and it does not depend on y and z in each cross section $x = \mathrm{const}$.

Let us present the results of calculating the flux of angular momentum through the plane $x = \pm L$ lying in the Fresnel zone ($kL \gg 1$, $k\Sigma \gg L$). In this zone $\mathbf{E}_d$ amounts to the field of a beam of the electromagnetic wave of cross section Σ and amplitudes $\mathbf{E}_{d\pm}$ (plus and minus signs respectively for $x > 0$ and $x < 0$), which are constant over the cross section of the beam everywhere except for a narrow boundary. We have the following expressions for the quantity G_{xx}:

$$G_{xx} = \frac{\Sigma}{8\pi k}\,\mathrm{Im}\,[\mathbf{E}_{d-}\mathbf{E}_{d-}^*], \qquad x<0, \qquad (26)$$

$$= -\frac{\Sigma}{8\pi k}\,\mathrm{Im}\,[\mathbf{E}_{d+}\mathbf{E}_{d+}^*] - \frac{\Sigma}{4\pi k}\,\mathrm{Im}\,[\mathbf{E}_{d+}\mathbf{E}_{\omega 0}^*], \quad x>0. \qquad (27)$$

The latter were derived from (11) after expressing the contributions to the quantities $\mathbf{E}_d$ and $\mathbf{B}_d$ from all the volume elements of the plate and integrating over the volume of the plate and over the planes $x = \pm L$ by using the stationary-phase method. The calculations prove to be lengthy owing to the high multiplicity of the integration, and hence have been omitted.

The reflected beam of waves ($x < 0$) bears away the angular momentum flux of (26). For a wave with circular polarization the quantity G_{xx} evidently differs from zero and moreover is associated with the energy flux $S_x\Sigma$ by the relationship

$$G_{xx} = \pm\,S_x\Sigma/\omega\,, \qquad (28)$$

which again is analogous to (1).

We arrive at the important conclusion that a beam with a finite cross section transports a finite flux of angular momentum G_{xx}, despite the fact that the field of the beam almost throughout the cross section coincides with the field of a plane wave and $g_{xx} = 0$. Nevertheless, in the narrow edge of the beam the wave is not plane, and the Poynting vector $\mathbf{S}$ is not necessarily parallel to the axis of the beam. Precisely in this zone the flux density g_{xx} differs from zero (which does not contradict (12)), and integrating it yields a finite value of G_{xx}. Representation of (26) in the form $G_{xx} = \Sigma[\mathrm{Im}\,\mathbf{E}_{d-}, \mathrm{Re}\,\mathbf{E}_{d-}]/(4\pi k)$ helps in understanding that $G_{xx} = 0$ for a linearly polarized wave (the vector $\mathrm{Im}\,\mathbf{E}_{d-}$ is parallel to $\mathrm{Re}\,\mathbf{E}_{d-}$), and that G_{xx} is maximal in magnitude for a beam with circular polarization ($\mathrm{Im}\,\mathbf{E}_{d-}$ is perpendicular to $\mathrm{Re}\,\mathbf{E}_{d-}$).

Further we assume that the reflection coefficient from the plate equals zero (because the permittivity near the edge of the plate approaches unity). Then G_{xx} $(x<0)=0$. The torque acting on the plate here equals $dM_x/dt = -G_{xx}$ $(x>0)$, and

$$\frac{dM_x}{dt} = \frac{\Sigma}{8\pi k}\,\mathrm{Im}\,[(\mathbf{E}_{d+}+\mathbf{E}_{w0})(\mathbf{E}^*_{d+}+\mathbf{E}^*_{w0})] - \frac{\Sigma}{8\pi k}\,\mathrm{Im}\,[\mathbf{E}_{w0}\mathbf{E}^*_{w0}]. \tag{29}$$

In discussing Eq. (29) we note the following. First, if we were to start only from the final relationship (29) without taking account of (26) and (27), as was actually done in Ref. 4, the statement in that paper would seem natural that the flux density of angular momentum in a plane wave, in accord with (1) and (2), equals $\mathrm{Im}[\mathbf{E}^*\mathbf{E}]/8\pi k$. Here the total fluxes in the cross section Σ would equal: $G_{xx}(x<0) = -\Sigma\,\mathrm{Im}[\mathbf{E}_{w0}\mathbf{E}^*_{w0}]/(8\pi k)$ and $G_{xx}(x>0) = \Sigma\,\mathrm{Im}[(\mathbf{E}_{d+}+\mathbf{E}_{w0})(\mathbf{E}^*_{d+}+\mathbf{E}^*_{w0})]/8\pi k$, and their difference would yield Eq. (29). But actually, as Eqs. (26) and (27) show, the distribution of the angular momentum flux is quite different.

Second, according to (29), we can interpret the quantity G_{xx} $(x>0)$ as the sum of the flux G_b in the beam transmitted through the plate (Fig. 1; the quantity G_b depends only on the amplitude of the transmitted wave $\mathbf{E}_b = \mathbf{E}_d + \mathbf{E}_{w0}$), and on the contribution from the "hole" in the front of the wave diffracted by the plate, which depends only on $\mathbf{E}_{w0}$:

$$G_{xx}(x>0) = G_b + G_h,$$

$$G_b = \frac{\Sigma}{8\pi k}\,\mathrm{Im}\,[\mathbf{E}^*_b\mathbf{E}_b],\quad G_h = \frac{\Sigma}{8\pi k}\,\mathrm{Im}\,[\mathbf{E}_{w0}\mathbf{E}^*_{w0}]. \tag{30}$$

However, this interpretation makes sense only in the limiting cases when G_b or G_h equals zero. For example, if the incident wave is absent ($\mathbf{E}_{w0}=0$) or is linearly polarized, then the angular momentum flux equals G_b and is concentrated in the beam of waves beyond the plate. Conversely, for total absorption of the wave in the plate, or if the transmitted wave acquires a pure linear polarization, then $G_b = 0$, and the angular momentum flux is localized in the edge of the wave diffracted by the plate. Yet in the general case when $G_b \neq 0$ and $G_h \neq 0$, the separation of the quantity G_{xx} $(x>0)$ into G_b and G_h is purely arbitrary, since the entire flux of angular momentum is actually localized in the narrow transition zone between the wave transmitted through the plate and the wave passing around the plate (see Fig. 1), rather than in either of these waves separately.

We easily note that, if the beam transmitted through the plate is circularly polarized, then the quantity G_b again is associated with the energy flux in the beam $S_x\Sigma$ by Eq. (28), which is analogous to the quantum relationship (1). The same equation relates the flux of angular momentum G_h in the "hole" with the deficiency of energy flux in this "hole."

We stress further that the main methodological difficulty in the question under discussion is associated precisely with the finite magnitude of G_h for an elliptically polarized wave: since G_h does not equal zero and does not approach zero as $k\to\infty$, the natural definition of the flux density of angular momentum of the electromagnetic field g_{xx} as the angular momentum acquired by a unit area upon total absorption of the waves incident on it proves to be impossible.

Finally, it is interesting to note that Sadovskiĭ's only remark on the angular momentum of the field pertains to a circularly polarized light beam that is formed upon passage of a linearly polarized beam through a $\lambda/4$ plate; the width of the plate is large in comparison with the diameter of the beam. Here a torque acts on the plate, and the natural assumption of the absence of angular momentum in the linearly polarized beam leads to the necessity of removal by the angular momentum of the beam of waves with circular polarization. This discussion is correct (if we overlook the ideas on the rotation of ether particles presented in this connection in Ref. 6). However, evidently is it valid precisely for a finite beam and bears no relation to the later incorrect conclusions[4] on the density of angular momentum in a plane wave.

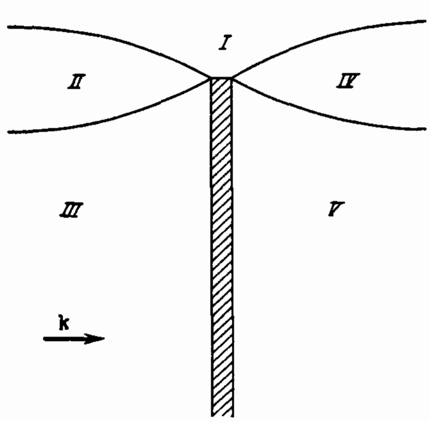

FIG. 1. Edge waves in the diffraction of a plane electromagnetic wave by a plate. The reflected wave (in region *III*) and the transmitted wave (in region *V*) are separated by the diffraction transition zones *II* and *IV* from region *I*, where only the field of the incident wave is present. **k** is the wave vector of the incident wave. The flux of angular momentum is localized in regions *II* and *IV*.

9. THE SADOVSKIĬ EFFECT IN A PLASMA AND GENERATION OF QUASI-STEADY-STATE CURRENTS

Now let us examine the Sadovskiĭ effect in a plasma accompanying collisional absorption of a circularly polarized electromagnetic wave.

Let the wave $\mathbf{E}_w = \mathbf{E}_{w0}\exp(-i\omega t + ikx)$ be incident on a plasma that occupies a cylindrical volume whose axis is parallel to the x axis. When we take account of the collisions of electrons and ions (ei), the permittivity of the plasma contains a positive imaginary component:[17,18]

$$\varepsilon = 1 - \frac{\omega_{pe}^2}{\omega(\omega+i\nu)}. \tag{31}$$

Here ν is the effective rate of ei collisions, while ω_{pe} is the electronic plasma frequency: $\omega_{pe}^2 = 4\pi Ne^2/m$, the charge and mass of the electrons equal $-e$ $(e>0)$ and m, and N is the concentration of the plasma.

The imaginary component of the permittivity involves the power absorbed by the elementary volume of the plasma $dx\,dy\,dz$:

$$dQ = \frac{\omega}{8\pi}\,\mathrm{Im}\,\varepsilon\,|\mathbf{E}_{w0}|^2 dx\,dy\,dz \doteq \frac{\omega_{pe}^2 \nu}{8\pi\omega^2}\,|\mathbf{E}_{w0}|^2 dx\,dy\,dz\,. \tag{32}$$

We assume in (32) and below for simplicity that $\omega_{pe} \ll \omega$, $\nu \ll \omega$. Under these conditions, we can consider the field in the plasma in the first approximation to coincide with the field of the incident wave (provided only that the dimensions of the plasma volume are not very large) and assume that the amplitude of the electric field equals $\mathbf{E}_{w0}$.

The energy of the wave that has passed through the plasma is partially absorbed in it. Therefore, beyond the plasma layer the field of this wave is somewhat smaller than $|\mathbf{E}_{w0}|$, and in line with (29), an angular momentum must be transferred to the plasma.

At first glance this can seem remarkable, since in a homogeneous plasma the electrons move in circles in the field of a circularly polarized wave:

$$\mathbf{r}_e = \mathbf{r}_0 + \frac{e\mathbf{E}_w}{m\omega^2}\,. \tag{33}$$

Then a transfer of time-averaged angular momentum occurs neither to the electron from the field, nor to the ion from the electrons (in (33) $\mathbf{r}_e$ is the time-dependent radius vector of the electron, while $\mathbf{r}_0 = \mathrm{const}$ is the coordinate of the electron in the absence of a field). However, the role emphasized above of edge effects in transfer of momentum from the field to objects hints that we must take account of the effects at the boundary of the plasma, where its concentration N uniformly declines to zero.

Therefore, let us examine the case in which N is homogeneous (for simplicity) along the x axis, but in the yz cross section the function N depends on y and z and vanishes at the edge of the plasma. To remain within the framework of linear plasma electrodynamics, we assume that the radius of the electronic orbits defined according to (33) is small in comparison with the characteristic scale of variation of the function $N(y,z)$.

Assuming the concentration gradient to be small, yet finite, let us average over time the frictional force acting on the electron owing to ei collisions $\mathbf{f}_e = -m\nu\mathbf{v}_e$, where $\mathbf{v}_e = d\mathbf{r}_e/dt$ is the velocity of the electron. We must take account of the small variation along the trajectory of the electron of the rate ν of collisions of the electron with ions, which is proportional to the concentration N of ions, and use the standard method of averaging quantities bilinear in the complex amplitude. We have

$$\langle \mathbf{f}_e \rangle = -\frac{e\nu(\mathbf{r}_0)}{2\omega^2 N}\,\mathrm{Re}\,\{(\mathbf{E}\,\mathrm{grad}\,N)\mathbf{v}^*\}$$
$$= \frac{e^2\nu(\mathbf{r}_0)}{2m\omega^3 N}\,\mathrm{Im}\,\{\mathbf{E}_{w0}^*(\mathbf{E}_{w0}\,\mathrm{grad}\,N)\}\,. \tag{34}$$

For a wave with pure circular polarization

$$\langle \mathbf{f}_e \rangle = \pm\frac{e^2\nu(\mathbf{r}_0)}{4m\omega^3 N}\,|\mathbf{E}_{w0}|^2\left[\frac{\mathbf{k}}{k}\,\mathrm{grad}\,N\right], \tag{35}$$

the wave vector $\mathbf{k}$ lies along the x axis. The force $\langle \mathbf{f}_e \rangle$ is directed along the tangent to the line $\nu = \mathrm{const}$ and compels the electron to move along the closed contour $N = \mathrm{const}$. Here the radius vector of the electron averaged over the period rotates in the same direction as the vector of the electric field in the electromagnetic wave. The reason for the appearance of the force $\langle \mathbf{f}_e \rangle$ in an inhomogeneous plasma and its orientation with respect to the vector grad N are evident from Fig. 2.

Now let us calculate the mean force acting on an individual ion, assuming for simplicity that all the ions are singly charged and neglecting the motion of the ions in the field of the wave. Here we must take account of the fact that in an inhomogeneous plasma in the field of a wave the electron concentration does not fully coincide with the concentration of ions $N(y,z)$ (see the explanation in Fig. 2). Since all the electrons in the plasma revolve in the same way, the concentration of electrons at the point $\mathbf{r}_e$ equals their initial concentration (before turning on the field) at the point $\mathbf{r}_0$, so that $n_e(\mathbf{r}_e) = N(\mathbf{r}_0)$. Therefore the mean frictional force that is exerted by the electrons on an ion $\mathbf{f}_i = m\nu_i\mathbf{v}_e$ and for which the rate of collisions of the ion with the electrons ν_i is proportional to the electron concentration is expressed after averaging in the form

$$\langle \mathbf{f}_i \rangle = \frac{e^2\nu(\mathbf{r}_0)}{2m\omega^3 N(\mathbf{r}_0)}\,\mathrm{Im}\,\{\mathbf{E}_{w0}^*(\mathbf{E}_{w0}\,\mathrm{grad}\,N)\}\,. \tag{36}$$

Thus we have $\langle \mathbf{f}_i \rangle = \langle \mathbf{f}_e \rangle$. We note that the values averaged over the volume of $\langle \mathbf{f}_i \rangle$ and $\langle \mathbf{f}_e \rangle$ equal zero, as we see from (36) (otherwise a contradiction would arise, since $\int dx\,dy\,dz(\langle \mathbf{f}_i \rangle + \langle \mathbf{f}_e \rangle) = 0$).

The force $\langle \mathbf{f}_i \rangle$ is closely associated with the so-called nonlinear current (see, e.g., Ref. 19). Actually, it is evident from the derivation of Eq. (36) that $\langle \mathbf{r}_i \rangle$ is proportional to

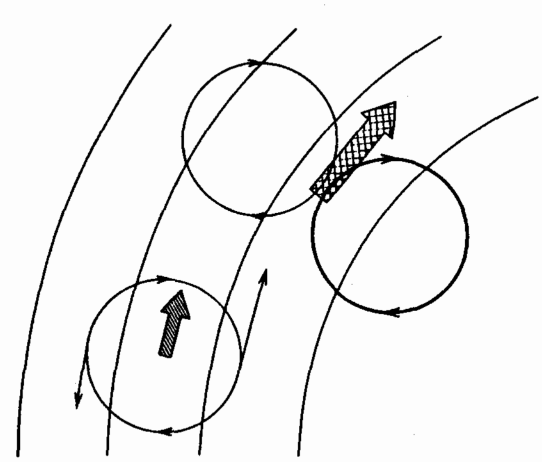

FIG. 2. Illustration of the reason for appearance of the average force of collisions in the field of a wave at the edge of a plasma. Here *0, 1, 2,* and *3* are lines of equal plasma density ($N = 0$, $N = 1$, etc.) expressed in certain arbitrary units; the circles with arrows on them show the trajectories and the direction of revolution of electrons. In the lower part of the diagram the thin arrows represent the vectors of the frictional force acting on an electron in different segments of its trajectory. Since the frictional force is proportional to the ion density, its magnitude is not constant, and the force averaged over the period (thick arrow) does not equal zero, and is directed along a line $N = \mathrm{const}$. In the upper part of the diagram the trajectories are arbitrarily drawn of electrons that act at different instants of time on a given ion. Here the thickness of the corresponding lines is related to the density of electrons moving along the given trajectory. Since the frictional force exerted on the ion by the electrons is proportional to the electron density averaged over the period of the force (thick arrow), it again differs from zero and is directed along a line $N = \mathrm{const}$.

the time-averaged product $\langle \delta n_e \mathbf{v}_e \rangle$, where δn_e is the change in the electron density at the fixed point $\mathbf{r}_0$ associated with the revolution of the electrons in the field. But the same mean contributes to the mean electron current $\mathbf{J}_n = -e\langle \delta N_e \mathbf{v}_e \rangle$:

$$\mathbf{J}_n = -\frac{e^3}{2m^2\omega^3} \operatorname{Im} \{\mathbf{E}^*_{w0}(\mathbf{E}_{w0} \operatorname{grad} N)\} . \quad (37)$$

One can say that the force $\langle \mathbf{f}_i \rangle = -m\nu \mathbf{J}_N/eN$ arises from the friction of the nonlinear current against the ions.

Integration of the bulk density of the torque $[\mathbf{R}_\perp \langle \mathbf{f}_e + \mathbf{f}_i \rangle]$ yields an expression for the total torque $d\mathbf{M}/dt$:

$$\frac{d\mathbf{M}}{dt} = -\frac{1}{2} \operatorname{Im} [\mathbf{E}_{w0}\mathbf{E}^*_{w0}] \int dx\, dy\, dz \frac{\nu\omega^2_{pe}}{4\pi\omega^3} = \frac{\operatorname{Im}[\mathbf{E}^*_{w0}\mathbf{E}_{w0}]}{\omega |\mathbf{E}_{w0}|^2} \int dQ \quad (38)$$

(see (32)), which fully agrees with the previous results. For circular polarization, as before, we have $d\mathbf{M}/dt = \pm \omega^{-1} \int dQ$.

Thus the absorption of a circularly polarized wave in a plasma leads to transfer of angular momentum to ions of the plasma. Let us stress to avoid misunderstanding that, despite the integral connection of (38) with the power absorbed in the bulk of the plasma (including the regions of uniform concentration), the transfer of angular momentum involves the forces acting only at the density gradients of the plasma (at the edge).

10. FEATURES OF THE INVERSE FARADAY EFFECT IN A PLASMA

The action of the force $\langle \mathbf{f}_e \rangle$ leads to dispersal of the electrons with respect to the ions. Here the mean velocity of the electrons $\langle \mathbf{v}_e \rangle$ reaches the steady-state value $\langle \mathbf{f}_e \rangle / m\nu$, which leads to the appearance of an entrainment current:

$$\mathbf{J}_c = -eN\langle \mathbf{v}_e \rangle = -\frac{e^3}{2m^2\omega^3} \operatorname{Im} \{\mathbf{E}^*_{w0}(\mathbf{E}_{w0} \operatorname{grad} N)\} = \mathbf{J}_n . \quad (39)$$

The quantity $\mathbf{J}_c$ does not depend on the rate of collisions (see the remark below), while the collisions determine only the time of establishment. In correspondence with the direction of the force $\langle \mathbf{f}_e \rangle$ the lines of electric current lie in the plane $x = \text{const}$ and coincide with the contours $N = \text{const}$ and hence are closed. We note that, on the one hand, the current in (39) is a special case of entrainment currents, and even Eq. (39) can be derived from the general formula for such currents (19). On the other hand, precisely the closed character of the currents in (39) for a circularly polarized wave is an important distinguishing feature, since usually the problem of closure of entrainment currents is not simple.[20]

Assuming for simplicity the plasma cylinder to be long (then one can neglect the dependence of the field on x), let us find the intensity of the quasistationary magnetic field. The Maxwell's equation and the equation of motion of the electrons for slowly varying quantities have the form:[21]

$$\partial\langle \mathbf{v}_e \rangle / \partial t = -(e/m) \langle \mathbf{E} \rangle - \nu_0 \langle \mathbf{v}_e \rangle + m^{-1} \langle \mathbf{f}_e \rangle, \quad \mathbf{J} = -eN\langle \mathbf{v}_e \rangle , \quad (40)$$

$$\operatorname{curl} \langle \mathbf{E} \rangle = -\frac{1}{c} \frac{\partial \langle \mathbf{B} \rangle}{\partial t} , \quad \operatorname{curl} \langle \mathbf{B} \rangle = \frac{4\pi}{c} (\mathbf{J} + \mathbf{J}_n) + \frac{1}{c} \frac{\partial \langle \mathbf{E} \rangle}{\partial t} . \quad (41)$$

In Eq. (40) the introduction of the different symbol ν_0 for the rate of collisions allows for the fact that in the relationship (31) for the permittivity the effective rate of collisions generally depends on ω (if only because of the dependence of the limits of cutoff of the Coulomb logarithm on ω). Therefore the quantity ν_0 in (40) that determines the low-frequency conductivity can differ from the quantity ν treated up to now, which is associated with the absorption of the plasma at the frequency ω (this distinction was neglected in (39)).

For short intervals of time $t \ll \nu_0^{-1}$ after turning on the field, the current $\mathbf{J}_n$ is compensated by the diamagnetic current $-eN\langle \mathbf{v}_e \rangle$ created by the induction electric fields, while the entrainment current has not set in. We note that above, when discussing the forces $\langle \mathbf{f}_e \rangle$ and $\langle \mathbf{f}_i \rangle$, the diamagnetic current was not taken into account. Nevertheless, taking account of the friction of this force against the ions in Eq. (38) would not alter the result (since the total force $\langle \mathbf{f}_e + \mathbf{f}_i \rangle$ does not depend on $\langle \mathbf{v}_e \rangle$), while in (40) the mean force acting on an electron is fully taken into account.

When $t \gg \nu_0^{-1}$, Eqs. (40) and (41) with account taken of the expressions for $\mathbf{J}_n$ and $\langle \mathbf{f}_e \rangle$ yield the equation of diffusion of the magnetic field

$$\frac{\omega^2_{pe}}{c^2\nu_0} \frac{\partial \langle \mathbf{B} \rangle}{\partial t} + \operatorname{curl} \operatorname{curl} \langle \mathbf{B} \rangle = \frac{4\pi}{c} \operatorname{curl} \left(\mathbf{J}_n + \frac{\nu}{\nu_0} \mathbf{J}_n\right) , \quad (42)$$

in which the role of the source of the field is played by the nonlinear current (the first term in the parentheses on the right-hand side) and the entrainment current, which differs when $\nu_0 \neq \nu$ from the previously derived expression (39) in the presence of the coefficient ν/ν_0. After the diffusion time the magnetic field reaches the steady-state value

$$\langle \mathbf{B} \rangle = \left(1 + \frac{\nu}{\nu_0}\right) \langle \mathbf{B}_n \rangle , \quad \langle \mathbf{B}_n \rangle = \frac{\pi e^3 N}{m^2 c \omega^3} \operatorname{Im} [\mathbf{E}_{w0}\mathbf{E}^*_{w0}] . \quad (43)$$

Outside the plasma the spatial distribution of the magnetic field coincides with the external field of a solenoid. The quantity $\langle \mathbf{B} \rangle$ depends on the rate of collisions only in terms of the ratio ν/ν_0 and does not depend on the magnitude nor the character of the edge concentration gradients that generate the entrainment current and the nonlinear current. For a fixed power of the electromagnetic wave, the field $\langle \mathbf{B} \rangle$ is strictly proportional to the degree of circular polarization, and in particular, changes sign upon replacing right-hand with left-hand polarization.

The generation of a steady-state magnetic field upon passing a circularly polarized wave through a medium is called the inverse Faraday effect.[22,23] The well known analysis of this effect for a transparent, nonmagnetic medium[22,23] is based on using an expression for the free energy in the presence of a magnetic field and a high-frequency wave, which can have a minimum at a nonzero intensity of the steady-state magnetic field if the high-frequency permittivity contains terms linearly dependent on $\langle \mathbf{B} \rangle$. However, the formal application of the resulting formulas[22,23] to a plasma, as was done, e.g., in Ref. 24, leads to the following expression (in the notation of Eq. (43)):

$$\langle \mathbf{B} \rangle = \langle \mathbf{B}_n \rangle , \quad (44)$$

This differs from (43) in the absence of the coefficient $1 + (\nu/\nu_0)$, or as is the same, in the neglect of the entrainment current as compared with the nonlinear current. Equation (44) has been derived[25] from other considerations, and again without taking account of the entrainment current.

However, there is no contradiction in this. The point is that one can treat a plasma as a transparent, nonmagnetic medium and apply to it Eq. (44) only in the case when, on the one hand, $\nu \to 0$ (otherwise the plasma is not fully transparent), or on the other hand, $\nu_0 \to \infty$ (otherwise the plasma is diamagnetic). But it is precisely when $\nu/\nu_0 \to 0$ that Eq. (43) converts to (44).

However, if the quantities ν and ν_0 are determined by a purely collisional mechanism, then $\nu \approx \nu_0$, the entrainment current equals the nonlinear current, and the magnetic field $\langle \mathbf{B} \rangle$ that is generated exceeds $\langle \mathbf{B}_n \rangle$ by approximately a factor of two.

Yet if the absorption of the electromagnetic wave is caused by any collective processes in the plasma,[26] then the magnitude of ν can increase by several orders of magnitude, and a case is possible in which $\nu \gg \nu_0$. Here, correspondingly, the magnetic field is generated mainly by the entrainment currents and exceeds $\langle \mathbf{B}_n \rangle$ by the ratio $\nu/\nu_0 \gg 1$. This remark may prove important since the generation of magnetic fields in the interaction of an electromagnetic wave with a plasma is studied experimentally over a very broad range of parameters, and apparently in a number of experiments the absorption of the wave is accompanied by turbulence of the plasma and increase in the effective magnitude of ν.

We note also that, after the establishment of the magnetic field of (43) throughout the volume of the plasma, the angular momentum absorbed in the plasma no longer increases the current but is fully transferred to the ions (the magnitude of $\langle \mathbf{f}_i \rangle$ increases twofold as compared with (36) owing to the friction of the entrainment current against the ions) and it causes a rotation of the plasma.

11. CONCLUSION

Despite the fact that the problems discussed in this article are mainly of methodological character, the results of the last two sections emphasize their close relation with the contemporary and pressing problems of plasma electrodynamics.

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